02: Combinatorics

Jerry Cain

April 3rd, 2024

Lecture Discussion on Ed

Summary of Combinatorics

Counting tasks on n objects

- Sort objects (permutations)
  - Distinct (distinguishable): $n!$
  - Some distinct
- Choose k objects (combinations)
- Put objects in r buckets

General approach to counting permutations

When there are n objects such that

n_1 are the same (indistinguishable or indistinct), and
n_2 are the same, and
...
n_r are the same,

The number of unique orderings (permutations) is

$$\frac{n!}{n_1!\, n_2! \cdots n_r!}.$$

For each group of indistinct objects,
divide by the overcounted permutations.

Sort semi-distinct objects

Order n semi-distinct objects $\frac{n!}{n_1!\,n_2!\cdots n_r!}$

How many permutations?

Strings

Order n semi-distinct objects	$\frac{n!}{n_1!\, n_2! \cdots n_r!}$

How many letter orderings are possible for the following strings?

1. KIKIIRIAFIN

2. EFFERVESCENCE

Strings

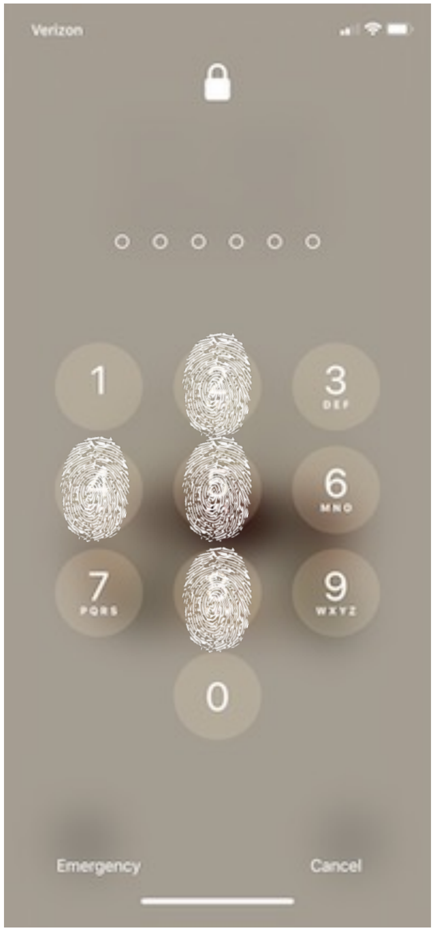
Order n semi-distinct objects $\frac{n!}{n_1!\,n_2!\cdots n_r!}$

How many letter orderings are possible for the following strings?

1. KIKIIRIAFIN $= \frac{11!}{5!2!} = 166{,}320$
2. EFFERVESCENCE $= \frac{13!}{2!5!2!} = 12{,}972{,}960$

Unique 6-digit passcodes with four smudges

Order n semi-distinct objects $\frac{n!}{n_1!\,n_2!\cdots n_r!}$



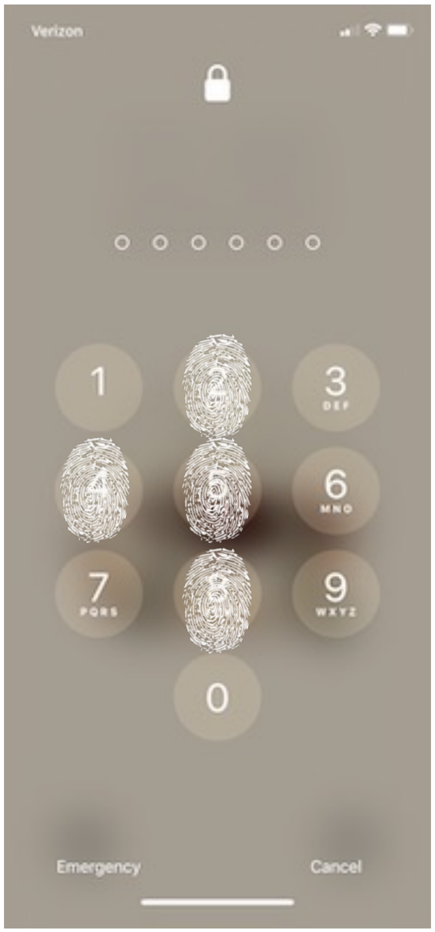
How many unique 6-digit passcodes are possible if a phone password uses each of four distinct numbers?

Two mutually exclusive scenarios:

- One digit repeated three times, other three repeated once
- Two digits repeated twice, other two repeated once

Unique 6-digit passcodes with four smudges

Order n semi-distinct objects $\frac{n!}{n_1!\,n_2!\cdots n_r!}$



How many unique 6-digit passcodes are possible if a phone password uses each of four distinct numbers?

Two mutually exclusive scenarios:

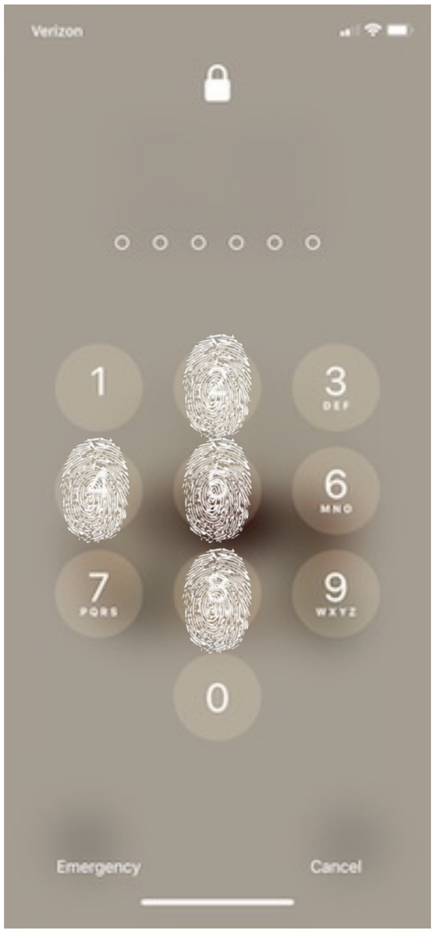
- One digit repeated three times, other three repeated once
- Two digits repeated twice, other two repeated once

first scenario: $n_1 = 4 \cdot \frac{6!}{3!} = 480$

second scenario: $n_2 = 6 \cdot \frac{6!}{2!2!} = 1080$

Unique 6-digit passcodes with four smudges

Order n semi-distinct objects $\frac{n!}{n_1!\,n_2!\cdots n_r!}$



How many unique 6-digit passcodes are possible if a phone password uses each of four distinct numbers?

Two mutually exclusive scenarios:

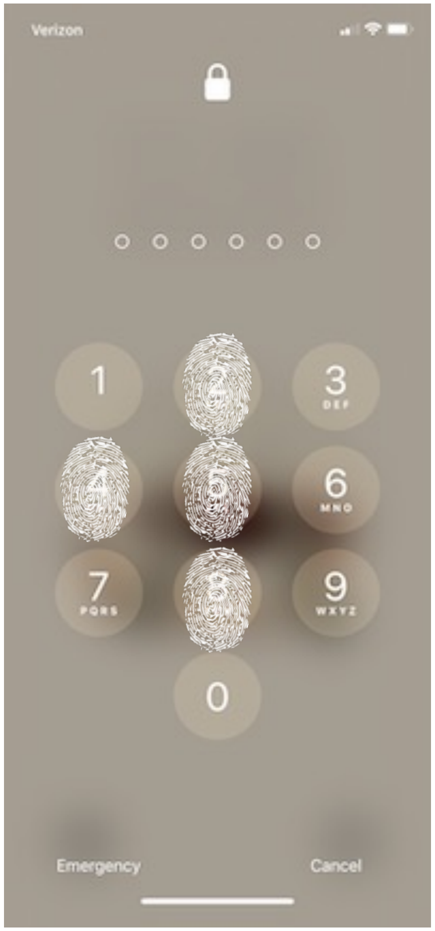
- One digit repeated three times, other three repeated once
- Two digits repeated twice, other two repeated once

first scenario: $n_1 = 4 \cdot \frac{6!}{3!} = 480$

second scenario: $n_2 = 6 \cdot \frac{6!}{2!2!} = 1080$

Unique 6-digit passcodes with four smudges

Order n semi-distinct objects $\frac{n!}{n_1!\,n_2!\cdots n_r!}$



How many unique 6-digit passcodes are possible if a phone password uses each of **four** distinct numbers?

Two mutually exclusive scenarios:

- One digit repeated three times, other three repeated once
- Two digits repeated twice, other two repeated once

first scenario: $n_1 = 4 \cdot \frac{6!}{3!} = 480$

second scenario: $n_2 = 6 \cdot \frac{6!}{2!2!} = 1080$

1560 such passcodes

Summary of Combinatorics

Counting tasks on n objects

- Sort objects (permutations)
  - Distinct (distinguishable): $n!$
  - Some distinct: $\frac{n!}{n_1!\,n_2!\cdots n_r!}$
- Choose k objects (combinations)
- Put objects in r buckets

Combinations I

Summary of Combinatorics

Counting tasks on n objects

- Sort objects (permutations)
  - Distinct (distinguishable): $n!$
  - Some distinct: $\frac{n!}{n_1!\,n_2!\cdots n_r!}$
- Choose k objects (combinations)
  - Distinct
- Put objects in r buckets

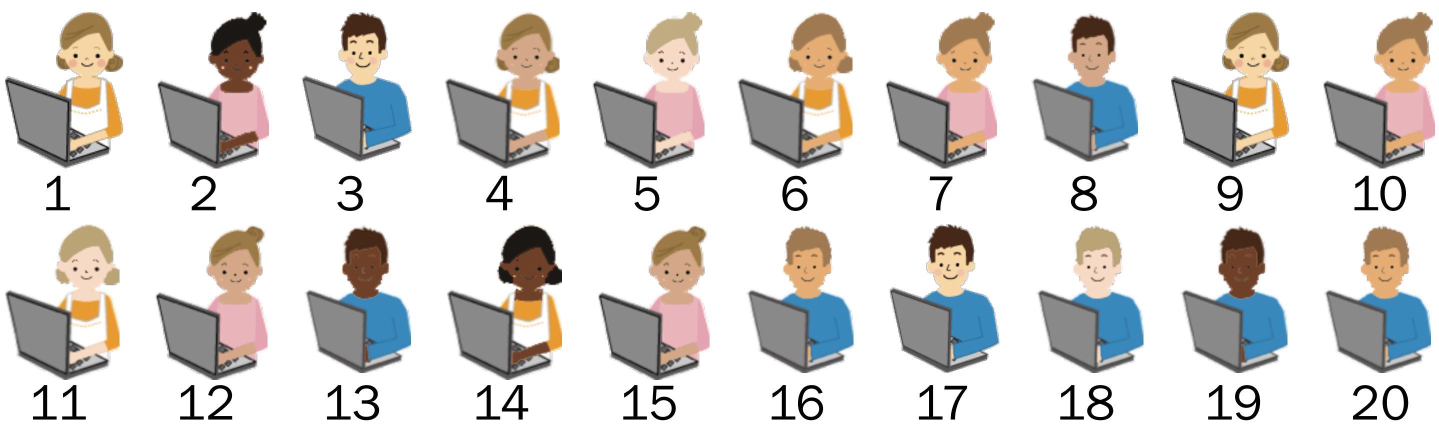
Combinations with cake

There are $n = 20$ people.
How many ways can we choose $k = 5$ people to get cake?

Consider the following generative process...

Combinations with cake

There are $n = 20$ people.
How many ways can we choose $k = 5$ people to get cake?

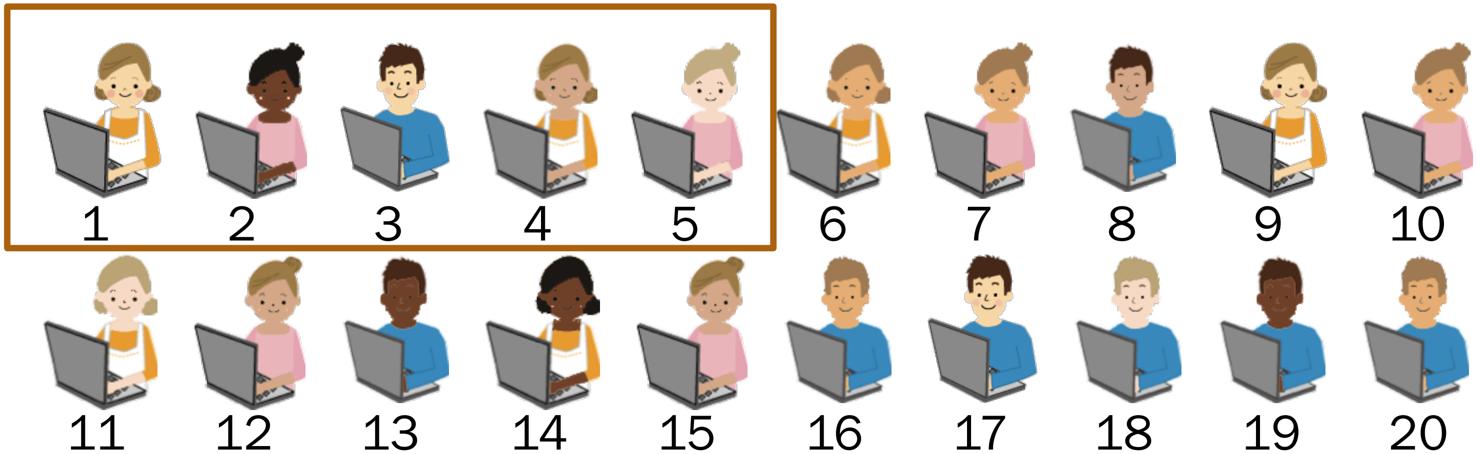


1. n people get in line

$n!$ ways

Combinations with cake

There are $n = 20$ people.
How many ways can we choose $k = 5$ people to get cake?



1. n people get in line

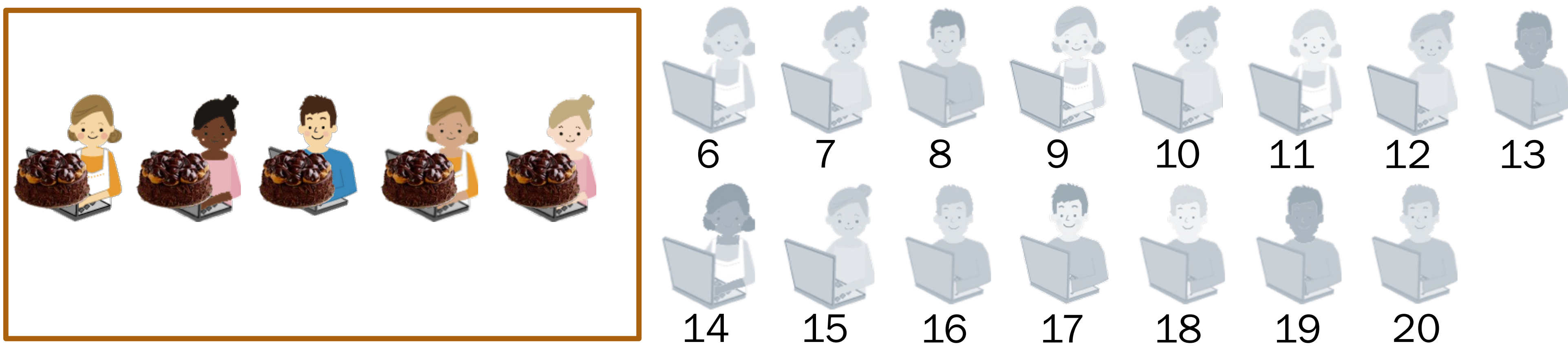
   $n!$ ways

2. Put first k in cake room

   1 way

Combinations with cake

There are $n = 20$ people.
How many ways can we choose $k = 5$ people to get cake?



1. n people get in line

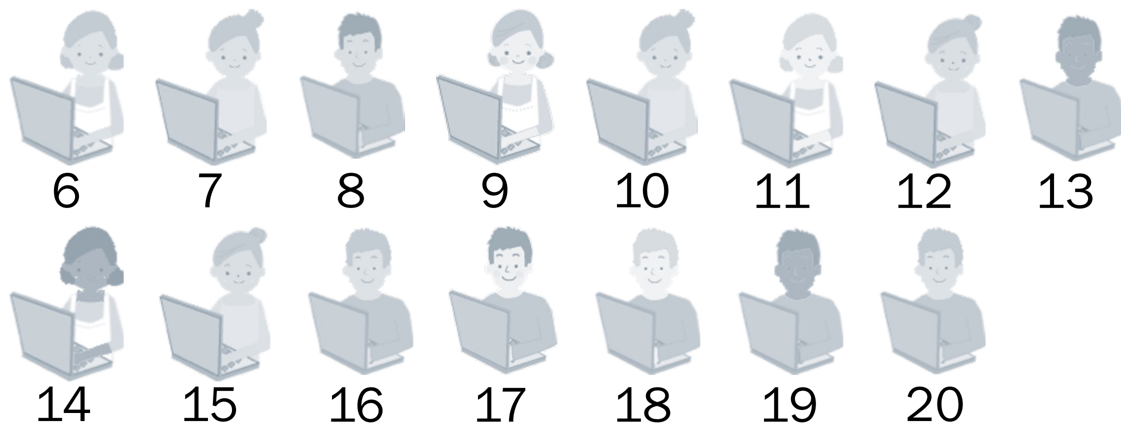
   $n!$ ways

2. Put first k in cake room

   1 way

Combinations with cake

There are $n = 20$ people.
How many ways can we choose $k = 5$ people to get cake?



1. n people get in line

   $n!$ ways

2. Put first k in cake room

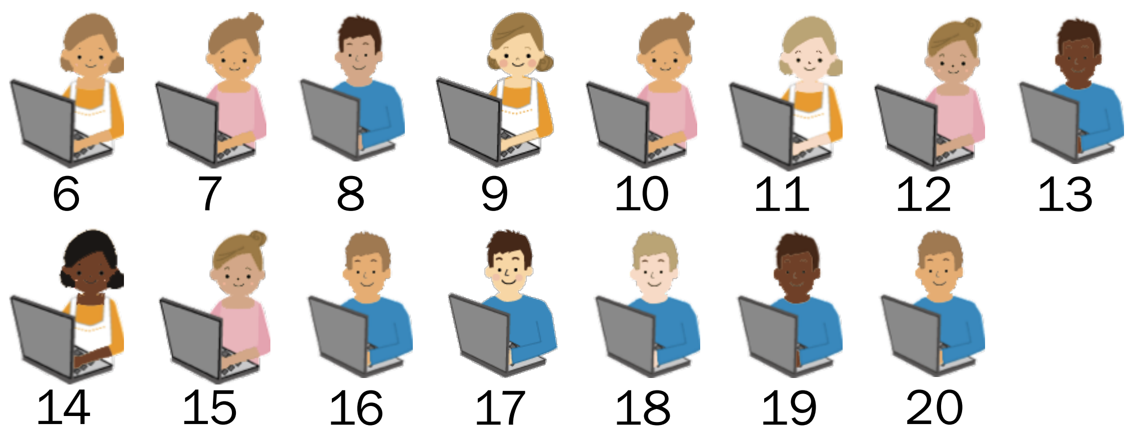
   1 way

3. Allow cake group to mingle

   $k!$ different permutations all considered the same group of children

Combinations with cake

There are $n = 20$ people.
How many ways can we choose $k = 5$ people to get cake?



1. n people get in line

   $n!$ ways

2. Put first k in cake room

   1 way

3. Allow cake group to mingle

   $k!$ different permutations all considered the same group of children

4. Allow non-cake group to mingle

Combinations with cake

There are $n = 20$ people.
How many ways can we choose $k = 5$ people to get cake?

1. n people get in line

   $n!$ ways

2. Put first k in cake room

   1 way

3. Allow cake group to mingle

   $k!$ different permutations all considered the same group of children

4. Allow non-cake group to mingle

   $(n-k)!$ different permutations all lead to the same group of children

Combinations

A combination is an unordered selection of k objects from a set of n distinct objects.

The number of ways of making this selection is

$$\frac{n!}{k!\,(n-k)!} = n! \times 1 \times \frac{1}{k!} \times \frac{1}{(n-k)!}$$

1. Order n distinct objects
2. Take first k as chosen
3. Overcounted: any ordering of chosen group is same choice
4. Overcounted: any ordering of unchosen group is same choice

Combinations

A combination is an unordered selection of k objects from a set of n distinct objects.

The number of ways of making this selection is

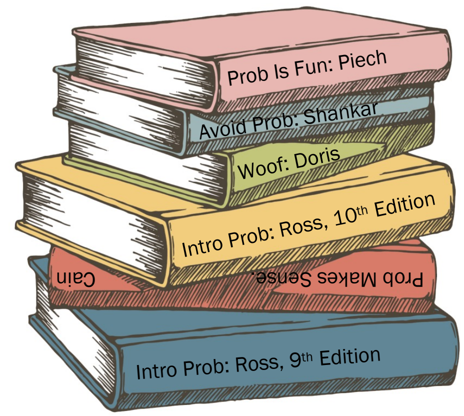
$$\frac{n!}{k!\,(n-k)!} = n! \times 1 \times \frac{1}{k!} \times \frac{1}{(n-k)!} = \binom{n}{k}$$

Binomial coefficient

Note: $\binom{n}{n-k} = \binom{n}{k}$

Probability textbooks

Choose k of n distinct objects $\binom{n}{k}$



How many ways are there to choose a **subset** of 3 from a set of 6 distinct books? By saying **subset**, we assume order doesn't matter.

$$\binom{6}{3} = \frac{6!}{3!\,3!} = 20 \text{ ways}$$

Combinations II

Summary of Combinatorics

Counting tasks on n objects

- **Sort objects (permutations)**
  - Distinct (distinguishable): $n!$
  - Some distinct: $\dfrac{n!}{n_1!\,n_2!\cdots n_r!}$
- **Choose k objects (combinations)**
  - Distinct
    - 1 group: $\binom{n}{k}$
    - r groups
- **Put objects in r buckets**

General approach to combinations

The number of ways to choose r groups of n distinct objects such that

For all $i = 1, \dots, r$, group i has size n_i, and

$\sum_{i=1}^{r} n_i = n$ (all objects are assigned), is

$$\frac{n!}{n_1!\, n_2! \cdots n_r!} = \binom{n}{n_1, n_2, \cdots, n_r}$$

Multinomial coefficient

Datacenters

Choose k of n distinct objects into r groups of size $n_1, \dots n_r$ $\binom{n}{n_1, n_2, \cdots, n_r}$

13 different computers are to be allocated to 3 datacenters as shown in the table:

How many different divisions are possible?

Datacenter	# machines
A	6
B	4
C	3

A. $\binom{13}{6,4,3} = 60{,}060$

B. $\binom{13}{6}\binom{7}{4}\binom{3}{3} = 60{,}060$

C. $6 \cdot 1001 \cdot 10 = 60{,}060$

D. A and B

E. All of the above

Datacenters

Choose k of n distinct objects into r groups of size $n_1, \dots n_r$ $\binom{n}{n_1, n_2, \cdots, n_r}$

13 different computers are to be allocated to 3 datacenters as shown in the table:

How many different divisions are possible?

Datacenter	# machines
A	6
B	4
C	3

A. $\binom{13}{6,4,3} = 60{,}060$

B. $\binom{13}{6}\binom{7}{4}\binom{3}{3} = 60{,}060$

C. $6 \cdot 1001 \cdot 10 = 60{,}060$

D. A and B

E. All of the above

Datacenters

Choose k of n distinct objects into r groups of size $n_1, \ldots n_r$ $\binom{n}{n_1, n_2, \cdots, n_r}$

13 different computers are to be allocated to 3 datacenters as shown in the table:

Datacenter	# machines
A	6
B	4
C	3

How many different divisions are possible?

A. $\binom{13}{6,4,3} = 60{,}060$

Strategy: Combinations into 3 groups

Group 1 (datacenter A): $n_1 = 6$

Group 2 (datacenter B): $n_2 = 4$

Group 3 (datacenter C): $n_3 = 3$

Datacenters

Choose k of n distinct objects into r groups of size $n_1, \ldots n_r$ $\binom{n}{n_1, n_2, \cdots, n_r}$

13 different computers are to be allocated to 3 datacenters as shown in the table:

How many different divisions are possible?

Datacenter	# machines
A	6
B	4
C	3

A. $\binom{13}{6,4,3} = 60{,}060$

Strategy: Combinations into 3 groups

Group 1 (datacenter A): $n_1 = 6$

Group 2 (datacenter B): $n_2 = 4$

Group 3 (datacenter C): $n_3 = 3$

B. $\binom{13}{6}\binom{7}{4}\binom{3}{3} = 60{,}060$

Strategy: Product rule with 3 steps

1. Choose 6 computers for A $\binom{13}{6}$
2. Choose 4 computers for B $\binom{7}{4}$
3. Choose 3 computers for C $\binom{3}{3}$

Datacenters

Choose k of n distinct objects into r groups of size $n_1, \dots n_r$ $\binom{n}{n_1, n_2, \cdots, n_r}$

13 different computers are to be allocated to 3 datacenters as shown in the table:

Datacenter	# machines
A	6
B	4
C	3

How many different divisions are possible?

A. $\binom{13}{6,4,3} = 60{,}060$

Strategy: Combinations into 3 groups

Group 1 (datacenter A): $n_1 = 6$

Group 2 (datacenter B): $n_2 = 4$

Group 3 (datacenter C): $n_3 = 3$

B. $\binom{13}{6}\binom{7}{4}\binom{3}{3} = 60{,}060$

Strategy: Product rule with 3 steps

1. Choose 6 computers for A $\binom{13}{6}$
2. Choose 4 computers for B $\binom{7}{4}$
3. Choose 3 computers for C $\binom{3}{3}$

Your approach will determine if you use binomial/multinomial coefficients or factorials.

Probability textbooks

Choose k of n distinct objects $\binom{n}{k}$

1. How many ways are there to choose 3 books from a set of 6 distinct books? $\binom{6}{3} = \frac{6!}{3!\,3!} = 20$ ways

2. Two are by the same author. What if we don't want to choose both?

A. $\binom{6}{3} - \binom{6}{2} = 5$ ways

B. $\frac{6!}{3!3!2!} = 10$

C. $2 \cdot \binom{4}{2} + \binom{4}{3} = 16$

D. $\binom{6}{3} - \binom{4}{1} = 16$

E. Both C and D

F. Something else

Probability textbooks

Choose k of n distinct objects $\binom{n}{k}$

1. How many ways are there to choose 3 books from a set of 6 distinct books? $\binom{6}{3} = \frac{6!}{3!\,3!} = 20$ ways

2. Two are by the same author. What if we don't want to choose both?

Strategy 1: Sum Rule

Probability textbooks

Choose k of n distinct objects $\binom{n}{k}$

1. How many ways are there to choose 3 books from a set of 6 distinct books?

$$\binom{6}{3} = \frac{6!}{3!\,3!} = 20 \text{ ways}$$

2. Two are by the same author. What if we don't want to choose both?

Strategy 2: "Forbidden method"

Forbidden method: It is sometimes easier to exclude invalid cases than to account for all valid cases.

Buckets and The Divider Method

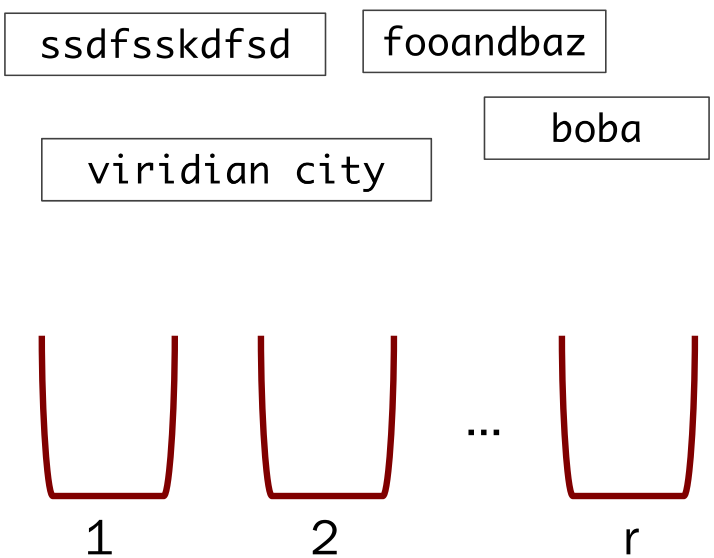
Summary of Combinatorics

Counting tasks on n objects

- **Sort objects (permutations)**
  - Distinct (distinguishable): $n!$
  - Some distinct: $\dfrac{n!}{n_1!\, n_2! \cdots n_r!}$
- **Choose k objects (combinations)**
  - Distinct
    - 1 group: $\binom{n}{k}$
    - r groups: $\binom{n}{n_1, n_2, \cdots, n_r}$
- **Put objects in r buckets**
  - Distinct
  - Indistinct

~~Balls and urns~~ Hash tables and distinct strings

How many ways are there to hash n distinct strings to r buckets?



Steps:

1. Bucket 1st string
2. Bucket 2nd string

...

n. Bucket nth string

r^n outcomes

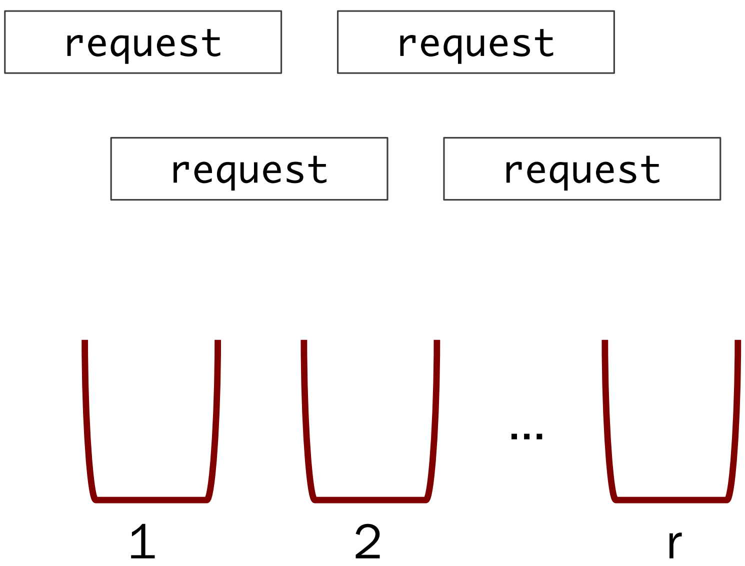
Summary of Combinatorics

Counting tasks on n objects

- **Sort objects (permutations)**
  - Distinct (distinguishable): $n!$
  - Some distinct: $\frac{n!}{n_1!\,n_2!\cdots n_r!}$
- **Choose k objects (combinations)**
  - Distinct
    - 1 group: $\binom{n}{k}$
    - r groups: $\binom{n}{n_1, n_2, \cdots, n_r}$
- **Put objects in r buckets**
  - Distinct: r^n
  - Indistinct

Servers and indistinct requests

How many ways are there to distribute n indistinct web requests to r servers?



Goal

Server 1 has x_1 requests,

Server 2 has x_2 requests,

…

Server r has x_r requests (the rest)

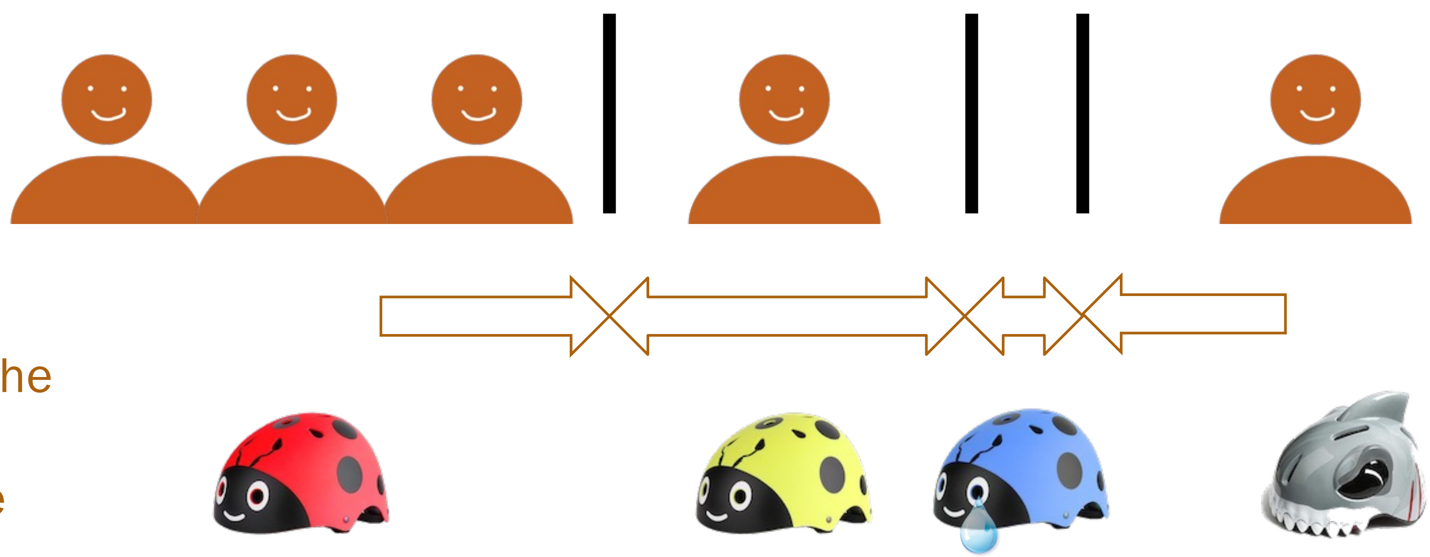
Bicycle helmet sales

How many ways can we assign $n = 5$ indistinct children to $r = 4$ distinct bicycle helmet styles?

Bicycle helmet sales

1 possible assignment outcome:

Goal Order n indistinct objects and $r - 1$ indistinct dividers.



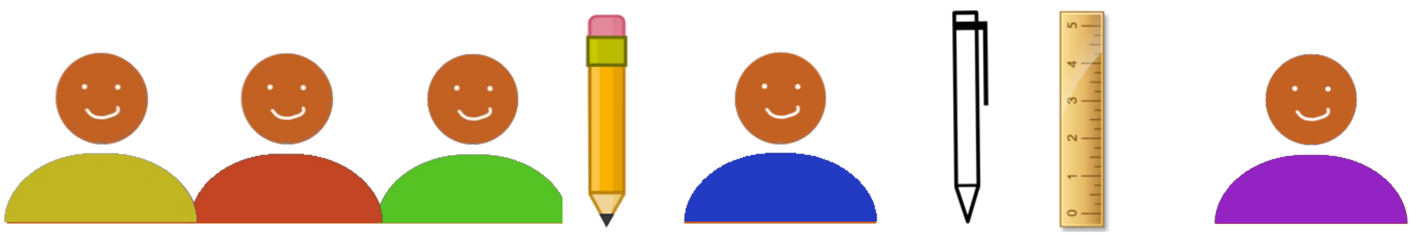
Consider the following generative process...

The divider method: A generative proof

How many ways can we assign $n = 5$ indistinct children to $r = 4$ distinct bicycle helmet styles?

Goal Order n indistinct objects and $r - 1$ indistinct dividers.

0. Make objects and dividers distinct

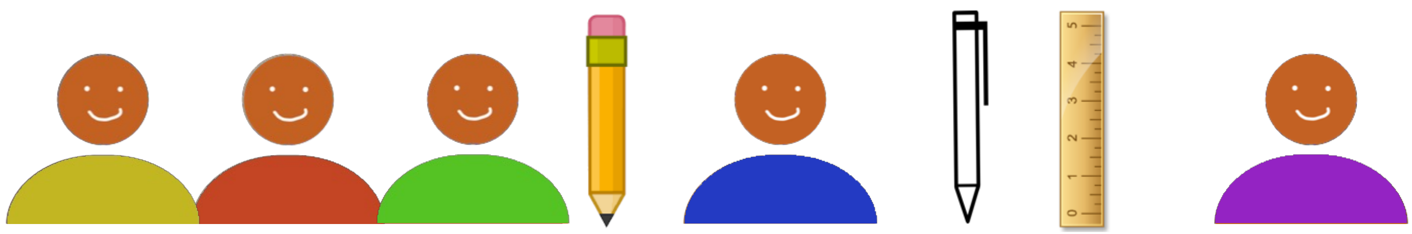


The divider method: A generative proof

How many ways can we assign $n = 5$ indistinct children to $r = 4$ distinct bicycle helmet styles?

Goal Order n indistinct objects and $r - 1$ indistinct dividers.

0. Make objects and dividers distinct



1. Order n distinct objects and $r - 1$ distinct dividers

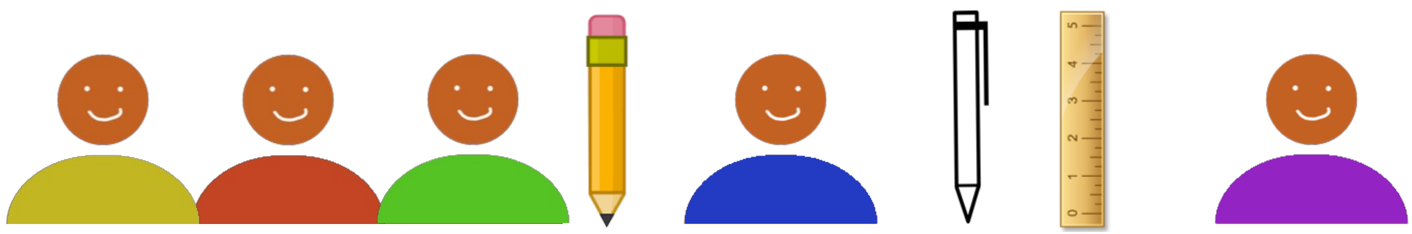
$$(n + r - 1)!$$

The divider method: A generative proof

How many ways can we assign $n = 5$ indistinct children to $r = 4$ distinct bicycle helmet styles?

Goal Order n indistinct objects and $r - 1$ indistinct dividers.

0. Make objects and dividers distinct



1. Order n distinct objects and $r - 1$ distinct dividers

$$(n + r - 1)!$$

2. Make n objects indistinct

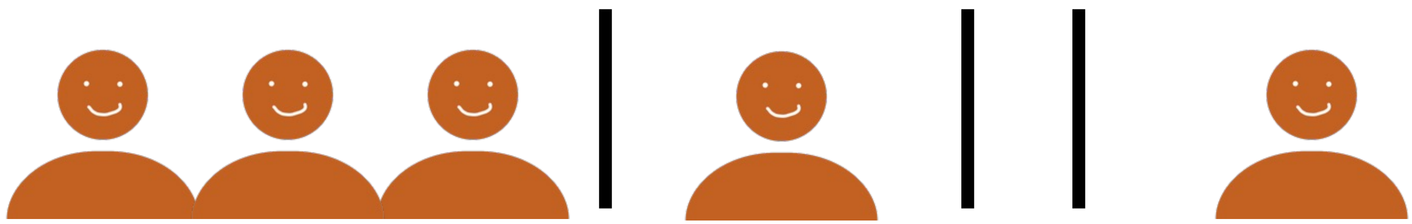
$$\frac{1}{n!}$$

The divider method: A generative proof

How many ways can we assign $n = 5$ indistinct children to $r = 4$ distinct bicycle helmet styles?

Goal Order n indistinct objects and $r - 1$ indistinct dividers.

0. Make objects and dividers distinct



1. Order n distinct objects and $r - 1$ distinct dividers

$$(n + r - 1)!$$

2. Make n objects indistinct

$$\frac{1}{n!}$$

3. Make $r - 1$ dividers indistinct

$$\frac{1}{(r-1)!}$$

The divider method

The number of ways to distribute n indistinct objects into r buckets is equivalent to the number of ways to permute $n + r - 1$ objects such that

n are indistinct objects, and

$r - 1$ are indistinct dividers:

$$\text{Total} = (n + r - 1)! \times \frac{1}{n!} \times \frac{1}{(r-1)!}$$

$$= \binom{n + r - 1}{r - 1} \text{ outcomes}$$

Venture capitalists

Divider method $\binom{n+r-1}{r-1}$
(n indistinct objects, r buckets)

You have $10 million to invest in 4 companies (in units of $1 million).

1. How many ways can you fully allocate your $10 million?
2. What if you want to invest at least $3 million in company 1?
3. What if you don't have to invest all your money?

Venture capitalists. #1

Divider method $\binom{n+r-1}{r-1}$
(n indistinct objects, r buckets)

You have $10 million to invest in 4 companies (in units of $1 million).

1. How many ways can you fully allocate your $10 million?

Set up

$$x_1 + x_2 + x_3 + x_4 = 10$$

x_i: amount invested in company i

$$x_i \geq 0$$

Solve

Venture capitalists. #2

Divider method $\binom{n+r-1}{r-1}$
(n indistinct objects, r buckets)

You have \$10 million to invest in 4 companies (in units of \$1 million).

1. How many ways can you fully allocate your \$10 million?
2. What if you want to invest at least \$3 million in company 1?

Set up

$$x_1 + x_2 + x_3 + x_4 = 10$$

x_i: amount invested in company i

Solve

Venture capitalists. #3

Divider method $\binom{n+r-1}{r-1}$
(n indistinct objects, r buckets)

You have \$10 million to invest in 4 companies (in units of \$1 million).

1. How many ways can you fully allocate your \$10 million?
2. What if you want to invest at least \$3 million in company 1?
3. What if you don't have to invest all your money?

Set up

$$x_1 + x_2 + x_3 + x_4 \leq 10$$

x_i: amount invested in company i

$$x_i \geq 0$$

Solve

Summary of Combinatorics

Counting tasks on n objects

- Sort objects (permutations)
  - Distinct (distinguishable)
  - Some distinct
- Choose k objects (combinations)
  - Distinct
    - 1 group
    - r groups
- Put objects in r buckets
  - Distinct
  - Indistinct

> - determine if objects are distinct
> - use product rule if several steps
> - use inclusion-exclusion if different cases

Combinatorial Proofs

Combinatorial Proofs

A **combinatorial proof**—sometimes called a **story proof**—is a proof that counts the same thing in two different ways, forgoing any tedious algebra.

Combinatorial proofs aren't as formal as CS103 proofs, but they still need to convince the reader something is true in an absolute sense.

An algebraic proof of, say, $\binom{n}{k} = \binom{n}{n-k}$ is straightforward if you just write combinations in terms of factorials.

A combinatorial proof makes an identity like $\binom{n}{k} = \binom{n}{n-k}$ easier to believe and understand *intuitively*.

Combinatorial Proof:

Consider choosing a set of k CS109 CAs from a total of n applicants. We know that there are $\binom{n}{k}$ such possibilities. Another way to choose the k CS109 CAs is to **disqualify** n – k applicants. There are $\binom{n}{n-k}$ ways to choose which n – k don't get the job. Specifying who is on CS109 course staff is the same as specifying who isn't. That means that $\binom{n}{k}$ and $\binom{n}{n-k}$ must be counting the same thing.

Combinatorial Proofs

Let's provide another combinatorial proof, this time proving that

$$n\binom{n-1}{k-1} = k\binom{n}{k}$$

This is easy to prove algebraically (provided k and n are positive integers, with $k \leq n$). A combinatorial/story proof, however, is more compelling!

Combinatorial Proof:

Consider n candidates for college admission, where k candidates can be accepted, and precisely one of the k is selected for a full scholarship. We can first choose the lucky recipient of the full scholarship and then select an additional k – 1 applicants from the remaining n – 1 applicants to round out the set of admits. Or we can select which k applicants are accepted and then choose which of those k gets the full ride.