02: Combinatorics

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Lecture Discussion on Ed

Summary of Combinatorics

relevant when order of objects matters

Counting tasks on n objects

- Sort objects (permutations)
  - Distinct (distinguishable): $n!$
  - Some distinct
- Choose k objects (combinations)
- Put objects in r buckets

relevant when order within selection does not matter

General approach to counting permutations

When there are n objects such that

n_1 are the same (indistinguishable or indistinct), and
n_2 are the same, and
...
n_r are the same,

The number of unique orderings (permutations) is

$$\frac{n!}{n_1!\,n_2!\cdots n_r!}.$$

simple example:
how many strings can be formed from the letters in:
MARRY

answer is: $\frac{5!}{1!\,1!\,2!\,1!} = 60$

For each group of indistinct objects,
divide by the overcounted permutations.

Sort semi-distinct objects

Order n semi-distinct objects $\frac{n!}{n_1!\,n_2!\cdots n_r!}$

How many permutations?

number of distinct orderings is $\frac{5!}{2!\,3!} = 10$

Strings

Order n semi-distinct objects $\frac{n!}{n_1!\,n_2!\cdots n_r!}$

How many letter orderings are possible for the following strings?

1. KIKIIRIAFIN

11 letters
2 K's, 5 I's, one of all others
$\frac{11!}{5!\,2!}$

2. EFFERVESCENCE

13 letters
2 F's, 5 E's, 2 C's, one of all others
$\frac{13!}{2!\,5!\,2!}$

Strings

Order n semi-distinct objects $\frac{n!}{n_1!\,n_2!\cdots n_r!}$

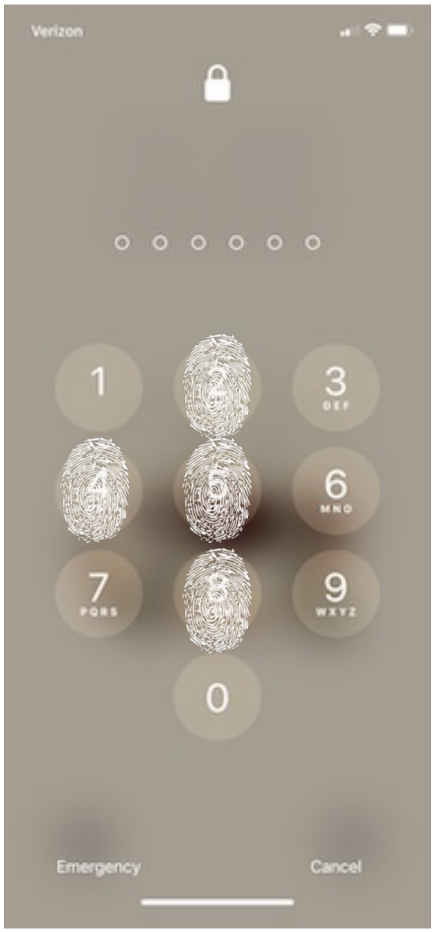
How many letter orderings are possible for the following strings?

1. KIKIIRIAFIN $= \frac{11!}{5!2!} = 166{,}320$

2. EFFERVESCENCE $= \frac{13!}{2!5!2!} = 12{,}972{,}960$

Unique 6-digit passcodes with four smudges

Order n semi-distinct objects $\frac{n!}{n_1!\,n_2!\cdots n_r!}$



How many unique 6-digit passcodes are possible if a phone password uses each of **four** distinct numbers?

Two mutually exclusive scenarios:

- One digit repeated three times, other three repeated once
- Two digits repeated twice, other two repeated once

Unique 6-digit passcodes with four smudges

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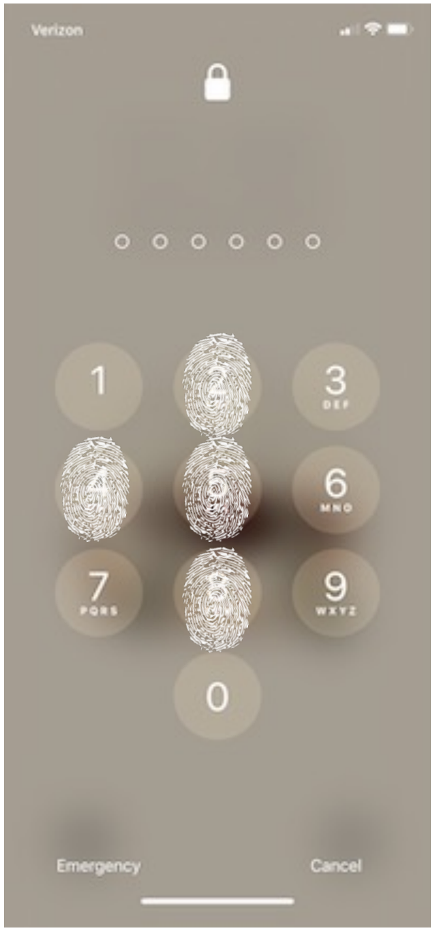

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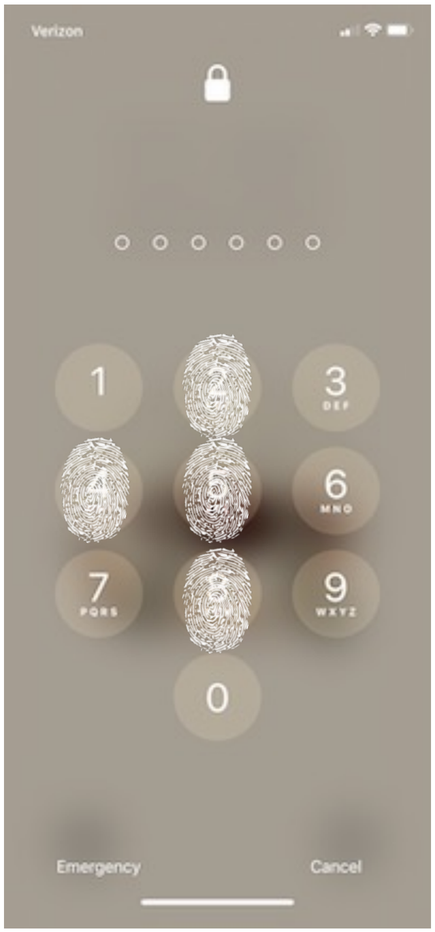
first scenario: $n_1 = 4 \cdot \frac{6!}{3!} = 480$

4 ways to choose the digit repeated three times

second scenario: $n_2 = 6 \cdot \frac{6!}{2!2!} = 1080$

Unique 6-digit passcodes with four smudges

Order n semi-distinct objects $\frac{n!}{n_1!\,n_2!\cdots n_r!}$



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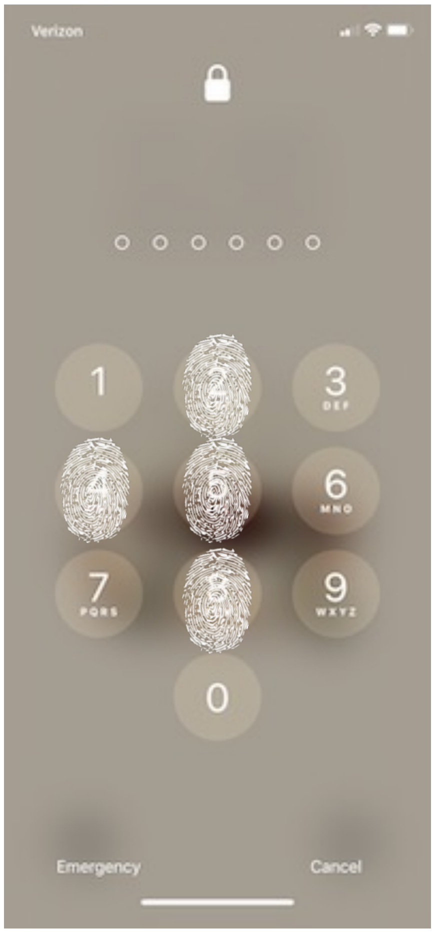
6 ways to choose two digits the each appear twice

24, 25, 28, 45, 48, 58

Unique 6-digit passcodes with four smudges

Order n semi-distinct objects $\frac{n!}{n_1!\,n_2!\cdots n_r!}$



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1560 such passcodes

Summary of Combinatorics

Counting tasks on n objects

- Sort objects (permutations)
  - Distinct (distinguishable): $n!$
  - Some distinct: $$\frac{n!}{n_1!\,n_2!\cdots n_r!}$$
- Choose k objects (combinations)
- Put objects in r buckets

Combinations I

Summary of Combinatorics

Counting tasks on n objects

- Sort objects (permutations)
  - Distinct (distinguishable): $n!$
  - Some distinct: $\dfrac{n!}{n_1!\,n_2!\cdots n_r!}$
- Choose k objects (combinations)
  - Distinct
- Put objects in r buckets

Combinations with cake

permutations care about order
combinations don't care about order

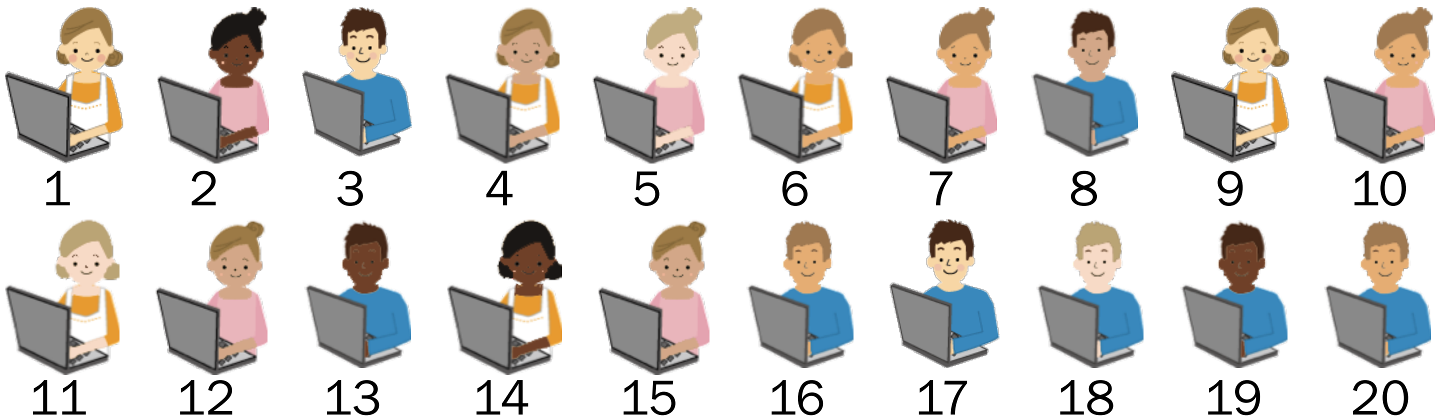
There are $n = 20$ people.
How many ways can we choose $k = 5$ people to get cake?

here, we don't order the children who get cake. they are not ranked. they are all peers!

Consider the following generative process…

Combinations with cake

There are $n = 20$ people.
How many ways can we choose $k = 5$ people to get cake?



1. n people get in line

$n!$ ways

Combinations with cake

There are $n = 20$ people.
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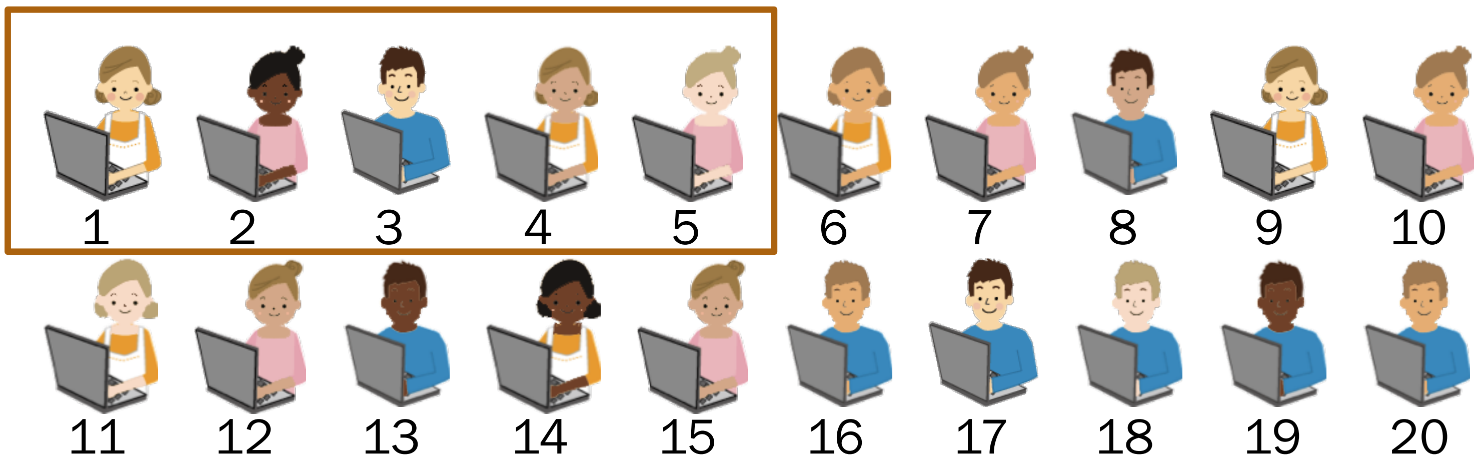

1. n people get in line

   $n!$ ways

2. Put first k in cake room

   1 way

Combinations with cake

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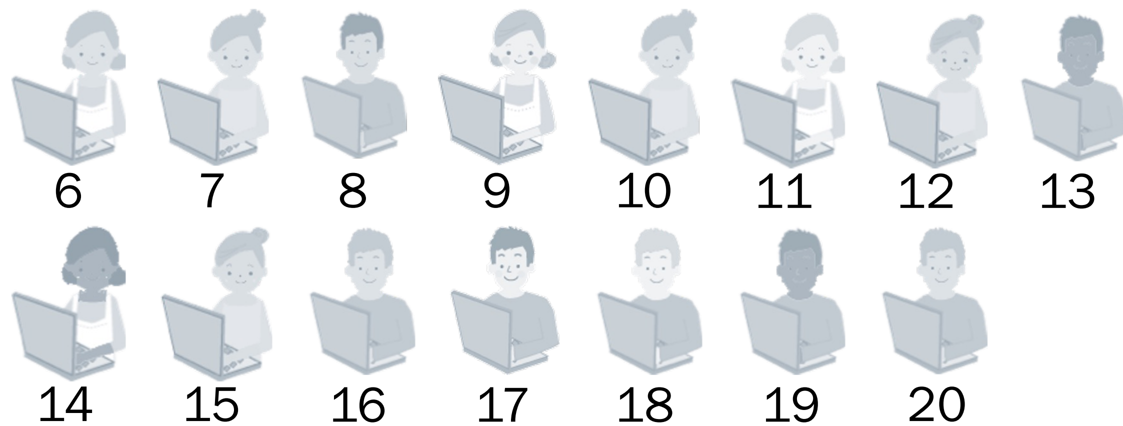

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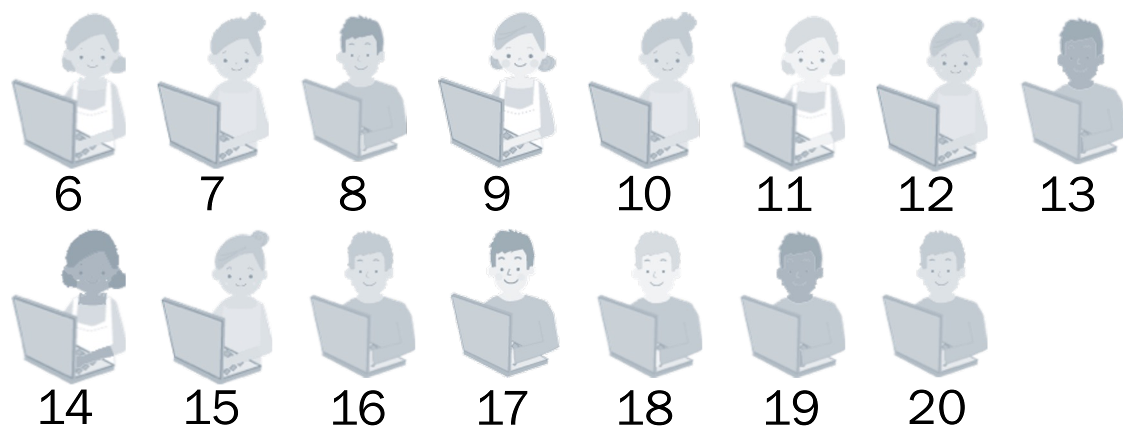

1. n people get in line

   $n!$ ways

2. Put first k in cake room

   1 way

3. Allow cake group to mingle

   $k!$ different permutations all considered the same group of children

Combinations with cake

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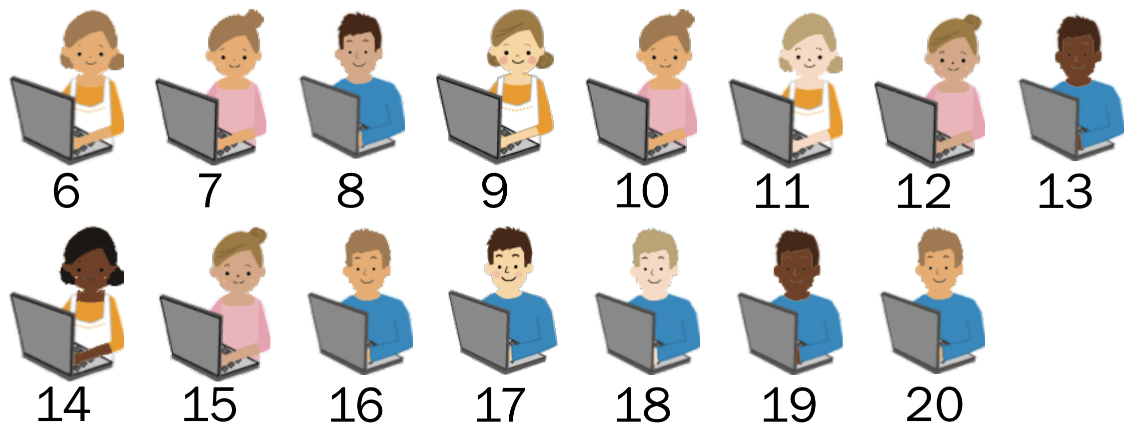

1. n people get in line

   $n!$ ways

2. Put first k in cake room

   1 way

3. Allow cake group to mingle

   $k!$ different permutations all considered the same group of children

4. Allow non-cake group to mingle

Combinations with cake

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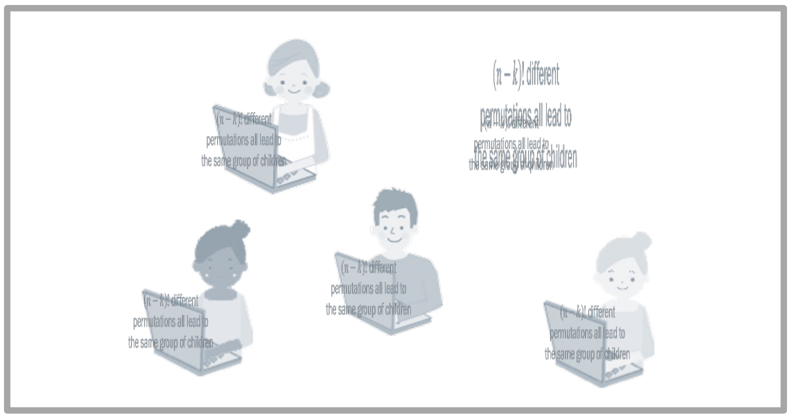


1. n people get in line

   $n!$ ways

2. Put first k in cake room

   1 way

3. Allow cake group to mingle

   $k!$ different permutations all considered the same group of children

4. Allow non-cake group to mingle

   $(n-k)!$ different permutations all lead to the same group of children

Combinations

A combination is an unordered selection of k objects from a set of n distinct objects.

The number of ways of making this selection is

$$\frac{n!}{k!\,(n-k)!} = n! \times 1 \times \frac{1}{k!} \times \frac{1}{(n-k)!}$$

1. Order n distinct objects
2. Take first k as chosen
3. Overcounted: any ordering of chosen group is same choice
4. Overcounted: any ordering of unchosen group is same choice

Combinations

A combination is an unordered selection of k objects from a set of n distinct objects.

The number of ways of making this selection is

read out loud as "n choose k"

$$\frac{n!}{k!\,(n-k)!} = n! \times 1 \times \frac{1}{k!} \times \frac{1}{(n-k)!} = \binom{n}{k}$$

Binomial coefficient

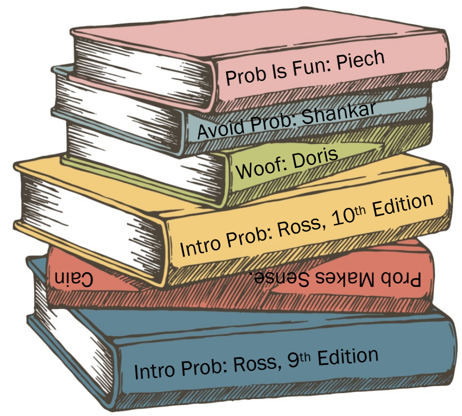
Note: $\binom{n}{n-k} = \binom{n}{k}$

the number of ways to select a group of 5 children from a class of 20 is "20 choose 5" $= \binom{20}{5} = \frac{20!}{5!\,15!} = 15504$

Probability textbooks

Choose k of n distinct objects $\binom{n}{k}$

How many ways are there to choose a **subset** of 3 from a set of 6 distinct books? By saying **subset**, we assume order doesn't matter.



$$\binom{6}{3} = \frac{6!}{3!\,3!} = 20 \text{ ways}$$

we don't care about the order of the three selected books

nor do we care about the order of those books we ignore.

Combinations II

Summary of Combinatorics

Counting tasks on n objects

- Sort objects (permutations)
  - Distinct (distinguishable): $n!$
  - Some distinct: $\frac{n!}{n_1!\, n_2! \cdots n_r!}$
- Choose k objects (combinations)
  - Distinct
    - 1 group: $\binom{n}{k}$
    - r groups
- Put objects in r buckets

General approach to combinations

The number of ways to choose r groups of n distinct objects such that

For all $i = 1, \dots, r$, group i has size n_i, and

$\sum_{i=1}^{r} n_i = n$ (all objects are assigned), is

$$\frac{n!}{n_1!\, n_2! \cdots n_r!} = \binom{n}{n_1, n_2, \cdots, n_r}$$

Multinomial coefficient

Datacenters

Choose k of n distinct objects into r groups of size $n_1, \ldots n_r$: $\binom{n}{n_1, n_2, \cdots, n_r}$

distinct, different, distinguishable — all synonyms

13 different computers are to be allocated to 3 datacenters as shown in the table:

How many different divisions are possible?

Datacenter	# machines
A	$n_A =$ 6
B	$n_B =$ 4
C	$n_C =$ 3

$n = 13$

A. $\binom{13}{6,4,3} = 60{,}060$

B. $\binom{13}{6}\binom{7}{4}\binom{3}{3} = 60{,}060$

C. $6 \cdot 1001 \cdot 10 = 60{,}060$

D. A and B

E. All of the above

Datacenters

Choose k of n distinct objects into r groups of size $n_1, \ldots n_r$ $\binom{n}{n_1, n_2, \cdots, n_r}$

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A. $\binom{13}{6,4,3} = 60{,}060$

Strategy: Combinations into 3 groups

Group 1 (datacenter A): $n_1 = 6$

Group 2 (datacenter B): $n_2 = 4$

Group 3 (datacenter C): $n_3 = 3$

$$\#\text{divisions} = \frac{13!}{6!\,4!\,3!} = 60{,}060$$

Datacenters

Choose k of n distinct objects into r groups of size $n_1, \dots n_r$ $\binom{n}{n_1, n_2, \cdots, n_r}$

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Group 3 (datacenter C): $n_3 = 3$

B. $\binom{13}{6}\binom{7}{4}\binom{3}{3} = 60{,}060$

Strategy: Product rule with 3 steps

1. Choose 6 computers for A $\binom{13}{6}$
2. Choose 4 computers for B $\binom{7}{4}$
3. Choose 3 computers for C $\binom{3}{3}$

$$\frac{13!}{6!\,7!} \cdot \frac{7!}{4!\,3!} \cdot \frac{3!}{3!\,0!} \Rightarrow \frac{13!}{6!\,4!\,3!} = \binom{13}{6,4,3}$$

Datacenters

Choose k of n distinct objects into r groups of size $n_1, \ldots n_r$ $\binom{n}{n_1, n_2, \cdots, n_r}$

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Strategy: Product rule with 3 steps

1. Choose 6 computers for A $\binom{13}{6}$
2. Choose 4 computers for B $\binom{7}{4}$
3. Choose 3 computers for C $\binom{3}{3}$

Your approach will determine if you use binomial/multinomial coefficients or factorials.

Probability textbooks

Choose k of n distinct objects $\binom{n}{k}$

1. How many ways are there to choose 3 books from a set of 6 distinct books? $\binom{6}{3} = \frac{6!}{3!\,3!} = 20$ ways

2. Two are by the same author. What if we don't want to choose both?

   A. $\binom{6}{3} - \binom{6}{2} = 5$ ways

   B. $\frac{6!}{3!3!2!} = 10$

   C. $2 \cdot \binom{4}{2} + \binom{4}{3} = 16$

   D. $\binom{6}{3} - \binom{4}{1} = 16$

   E. Both C and D

   F. Something else

Probability textbooks

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2. Two are by the same author. What if we don't want to choose both?

Strategy 1: Sum Rule

assume two of six books were written by Woolf
others by authors A, B, C, D

partition into three cases

Woolf	Woolf	A	B	C	D	
✓	✗	← choose any two →				$\Rightarrow \binom{4}{2}$
✗	✓	← choose any two →				$\Rightarrow \binom{4}{2}$
✗	✗	← choose three →				$\Rightarrow \binom{4}{3}$

answer is: $2\binom{4}{2} + \binom{4}{3} = 16$

Probability textbooks

Choose k of n distinct objects $\binom{n}{k}$

1. How many ways are there to choose 3 books from a set of 6 distinct books?

$$\binom{6}{3} = \frac{6!}{3!\,3!} = 20 \text{ ways}$$

2. Two are by the same author. What if we don't want to choose both?

answer is $\binom{6}{3} - \binom{4}{1} = 16$

Strategy 2: "Forbidden method"

count number of illegal subsets

Woolf ✓ Woolf ✓ A B C D

← choose just one → $\Rightarrow \binom{4}{1}$

Forbidden method: It is sometimes easier to exclude invalid cases than to account for all valid cases.

Buckets and The Divider Method

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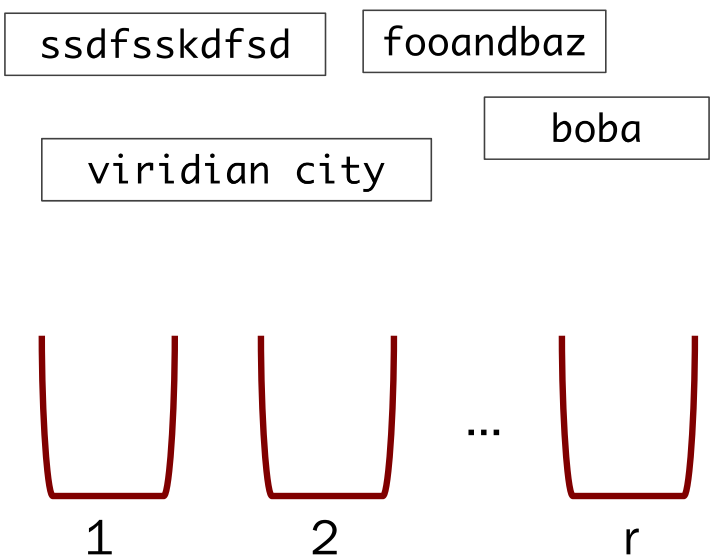
- **Choose k objects (combinations)**
  - Distinct
    - 1 group: $\binom{n}{k}$
    - r groups: $\binom{n}{n_1, n_2, \cdots, n_r}$
- **Put objects in r buckets**
  - Distinct
  - Indistinct

~~Balls and urns~~ Hash tables and distinct strings

How many ways are there to hash n distinct strings to r buckets?



Steps:

1. Bucket 1^{st} string → r choices
2. Bucket 2^{nd} string → r choices

...

n. Bucket n^{th} string → r choices

r^n outcomes

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- **Choose k objects (combinations)**
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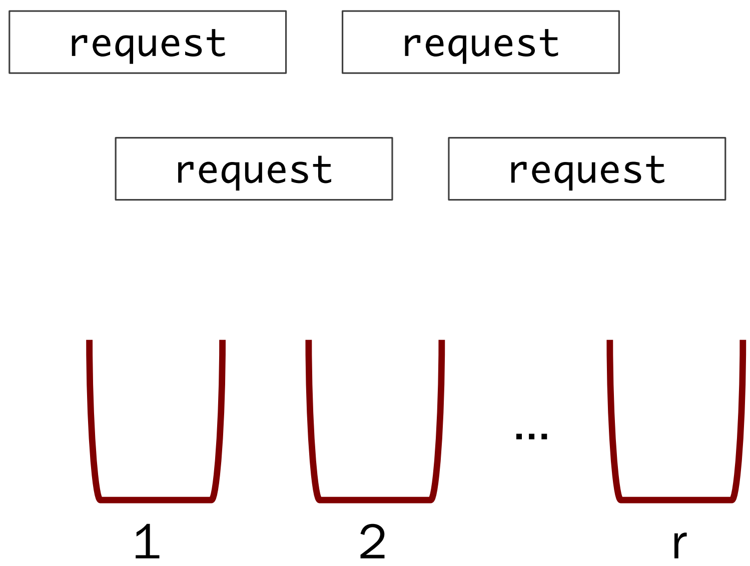
    - 1 group: $\binom{n}{k}$
    - r groups: $\binom{n}{n_1, n_2, \cdots, n_r}$
- **Put objects in r buckets**
  - Distinct: r^n
  - Indistinct

awesome, but tricky!!!

Servers and indistinct requests

How many ways are there to distribute n indistinct web requests to r servers?



Goal

Server 1 has x_1 requests,
Server 2 has x_2 requests,
…
Server r has x_r requests (the rest)

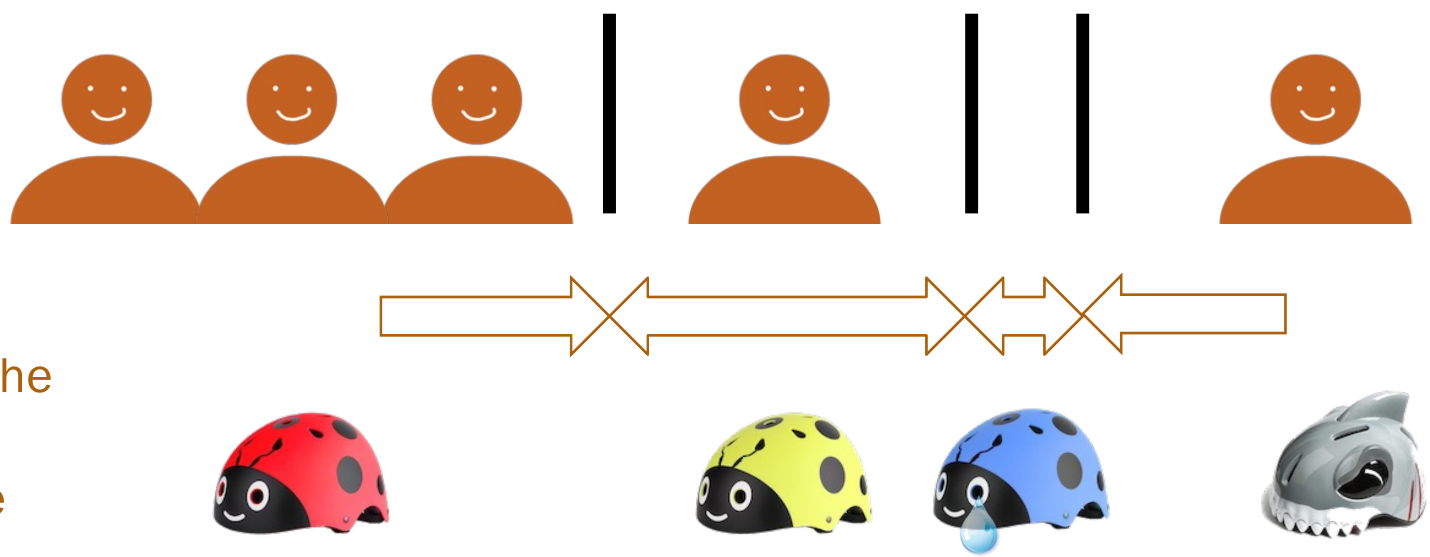
Bicycle helmet sales

How many ways can we assign $n = 5$ indistinct children to $r = 4$ distinct bicycle helmet styles?

Bicycle helmet sales

1 possible assignment outcome:

Goal Order n indistinct objects and $r - 1$ indistinct dividers.



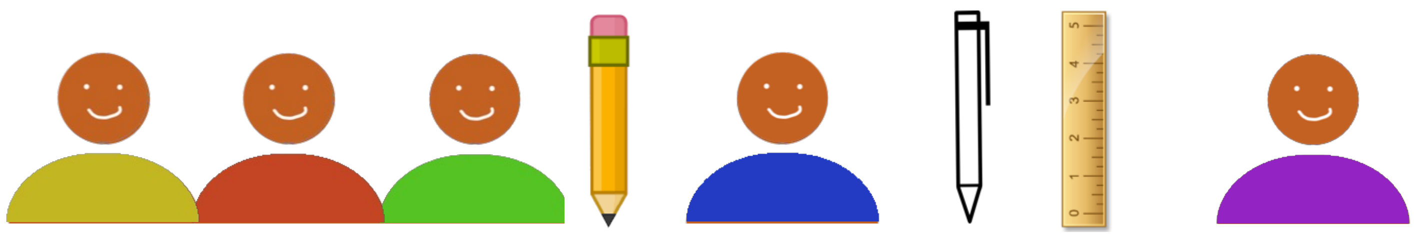
Consider the following generative process...

The divider method: A generative proof

How many ways can we assign $n = 5$ indistinct children to $r = 4$ distinct bicycle helmet styles?

Goal Order n indistinct objects and $r - 1$ indistinct dividers.

0. Make objects and dividers distinct



The divider method: A generative proof

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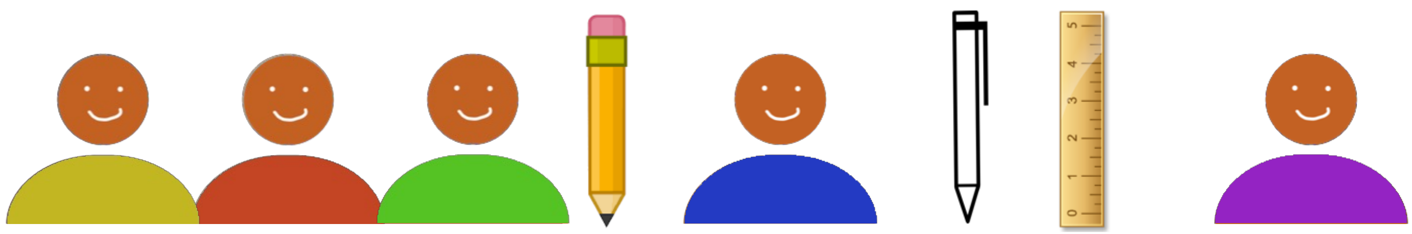

1. Order n distinct objects and $r - 1$ distinct dividers

$$(n + r - 1)!$$

The divider method: A generative proof

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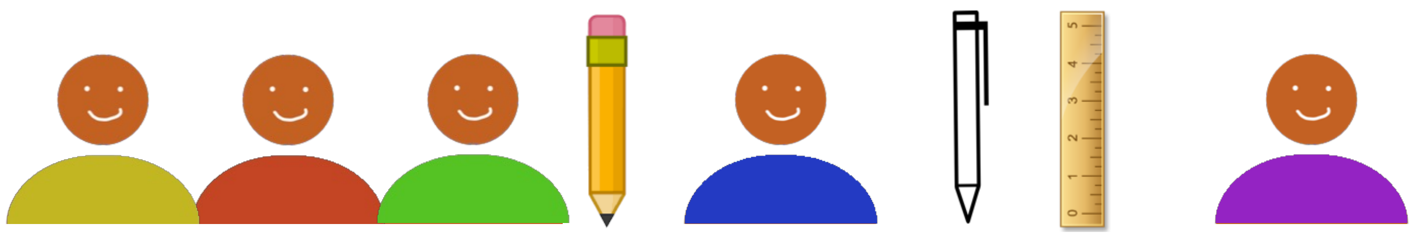

1. Order n distinct objects and $r - 1$ distinct dividers

$$(n + r - 1)!$$

2. Make n objects indistinct

$$\frac{1}{n!}$$

The divider method: A generative proof

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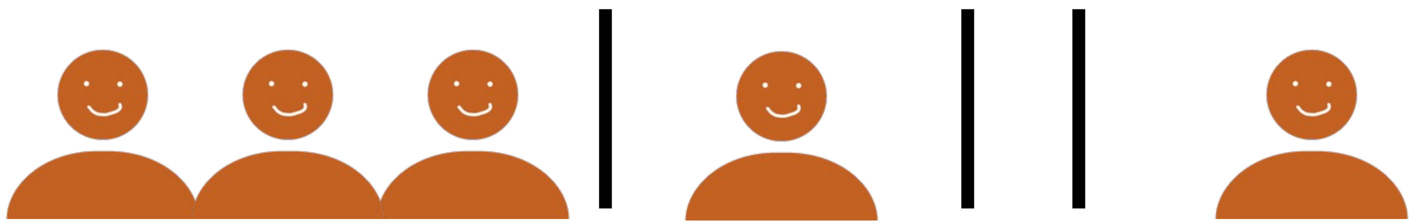


1. Order n distinct objects and $r - 1$ distinct dividers

$$(n + r - 1)!$$

2. Make n objects indistinct

$$\frac{1}{n!}$$

3. Make $r - 1$ dividers indistinct

$$\frac{1}{(r-1)!}$$

The divider method

buckets are distinct and ordered

The number of ways to distribute n indistinct objects into r buckets is equivalent to the number of ways to permute $n + r - 1$ objects such that

n are indistinct objects, and

$r - 1$ are indistinct dividers:

$$\text{Total} = (n + r - 1)! \times \frac{1}{n!} \times \frac{1}{(r-1)!}$$

$$= \binom{n + r - 1}{r - 1} \text{ outcomes}$$

Venture capitalists

Divider method
(n indistinct objects, r buckets) $\binom{n+r-1}{r-1}$

You have $10 million to invest in 4 companies (in units of $1 million).

1. How many ways can you fully allocate your $10 million?
2. What if you want to invest at least $3 million in company 1?
3. What if you don't have to invest all your money?

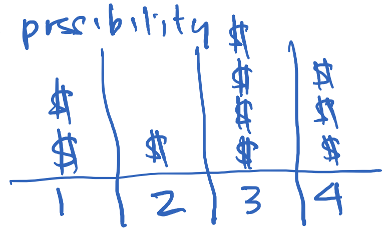
Venture capitalists. #1

Divider method (n indistinct objects, r buckets) $\binom{n+r-1}{r-1}$

You have \$10 million to invest in 4 companies (in units of \$1 million).

1. How many ways can you fully allocate your \$10 million?

one such possibility



Set up

$$x_1 + x_2 + x_3 + x_4 = 10$$

x_i: amount invested in company i

$$x_i \geq 0$$

x_i are integers

Solve

$$\binom{10+4-1}{4-1} = \binom{13}{3} = 286$$

Venture capitalists. #2

Divider method $\binom{n+r-1}{r-1}$
(n indistinct objects, r buckets)

You have $10 million to invest in 4 companies (in units of $1 million).

1. How many ways can you fully allocate your $10 million?
2. What if you want to invest at least $3 million in company 1?

Set up

$$x_1 + x_2 + x_3 + x_4 = 10$$

x_i: amount invested in company i

added constraint: $3M goes to company 1, but remaining $7M can be freely allocated

so, we're really solving $y_1 + y_2 + y_3 + y_4 = 7$, $y_i \geq 0$

Solve

$$\binom{7+4-1}{4-1} = \binom{10}{3} = 120$$

Venture capitalists. #3

Divider method $\binom{n+r-1}{r-1}$
(n indistinct objects, r buckets)

You have \$10 million to invest in 4 companies (in units of \$1 million).

1. How many ways can you fully allocate your \$10 million?
2. What if you want to invest at least \$3 million in company 1?
3. What if you don't have to invest all your money?

Set up ⚠️

$$x_1 + x_2 + x_3 + x_4 \leq 10$$

x_i: amount invested in company i

$$x_i \geq 0$$

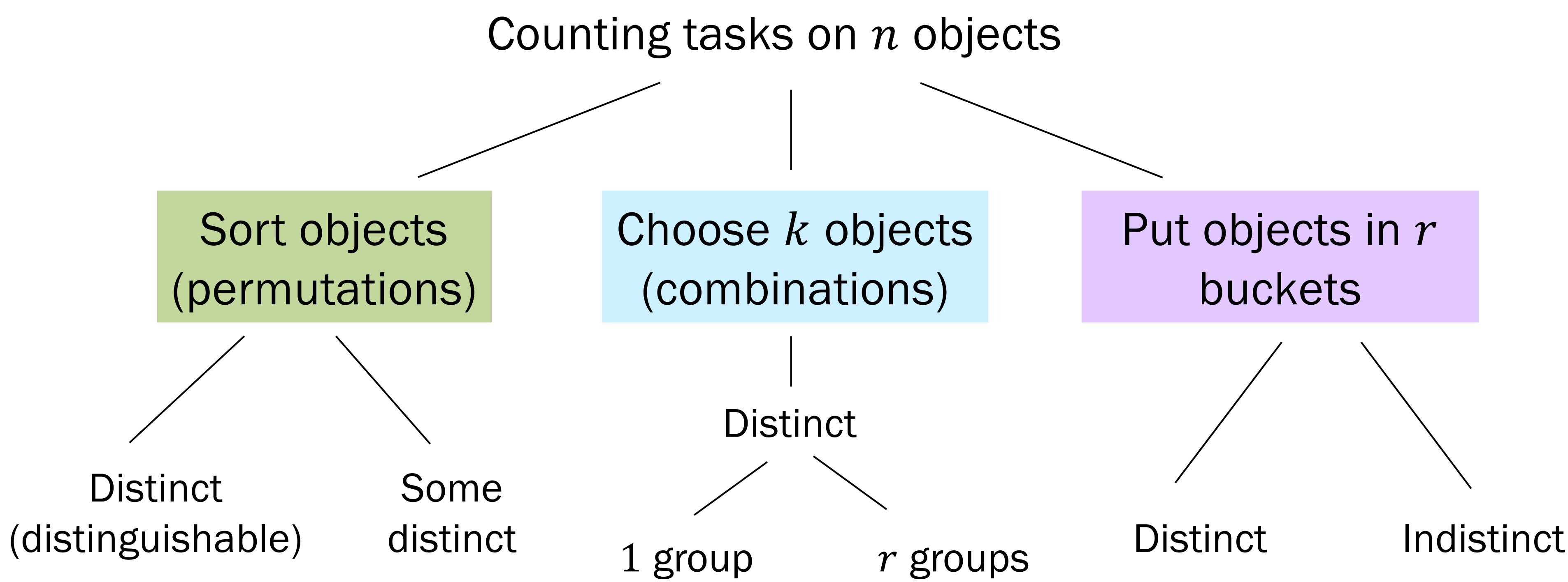
we are really solving
$x_1 + x_2 + x_3 + x_4 + x_5 = 10$,
where x_5 counts the money
you elect to not invest

Solve

$$\binom{10+5-1}{5-1} = \binom{14}{4}$$

$$= 1001$$

Summary of Combinatorics



- determine if objects are distinct
- use product rule if several steps
- use inclusion-exclusion if different cases

Combinatorial Proofs

Combinatorial Proofs

A **combinatorial proof**—sometimes called a **story proof**—is a proof that counts the same thing in two different ways, forgoing any tedious algebra.

Combinatorial proofs aren't as formal as CS103 proofs, but they still need to convince the reader something is true in an absolute sense.

An algebraic proof of, say, $\binom{n}{k} = \binom{n}{n-k}$ is straightforward if you just write combinations in terms of factorials.

A combinatorial proof makes an identity like $\binom{n}{k} = \binom{n}{n-k}$ easier to believe and understand *intuitively*.

Combinatorial Proof:

Consider choosing a set of k CS109 CAs from a total of n applicants. We know that there are $\binom{n}{k}$ such possibilities. Another way to choose the k CS109 CAs is to **disqualify** n – k applicants. There are $\binom{n}{n-k}$ ways to choose which n – k don't get the job. Specifying who is on CS109 course staff is the same as specifying who isn't. That means that $\binom{n}{k}$ and $\binom{n}{n-k}$ must be counting the same thing.

Combinatorial Proofs

Let's provide another combinatorial proof, this time proving that

$$n\binom{n-1}{k-1} = k\binom{n}{k}$$

This is easy to prove algebraically (provided k and n are positive integers, with $k \leq n$). A combinatorial/story proof, however, is more compelling!

Combinatorial Proof:

Consider n candidates for college admission, where k candidates can be accepted, and precisely one of the k is selected for a full scholarship. We can first choose the lucky recipient of the full scholarship and then select an additional k – 1 applicants from the remaining n – 1 applicants to round out the set of admits. Or we can select which k applicants are accepted and then choose which of those k gets the full ride.