03: Intro to Probability

Jerry Cain and Kanu Grover

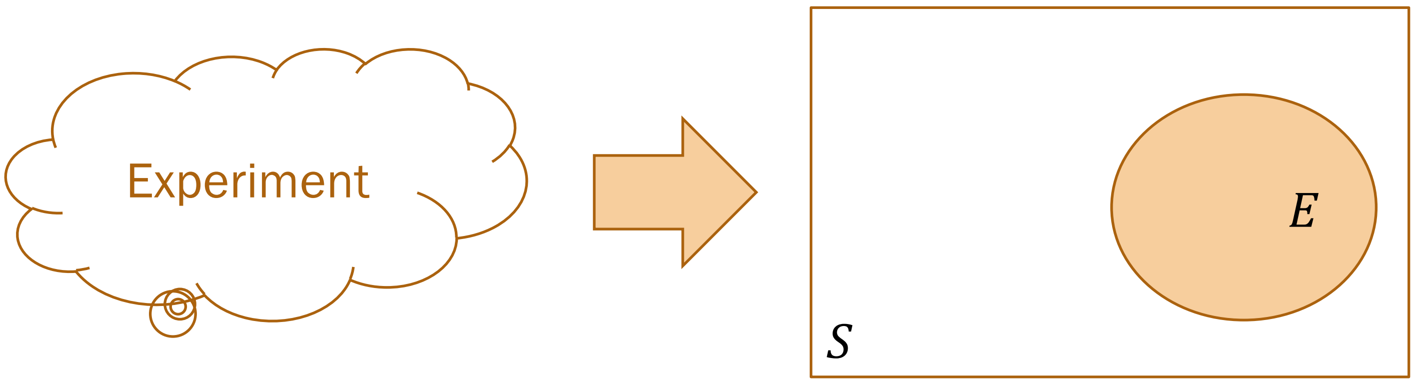
April 3rd, 2024

Lecture Discussion on Ed

Defining Probability

Key definitions

An experiment in probability:



Sample Space, S: The set of all possible outcomes of an experiment

Event, E: Some subset of S ($E \subseteq S$).

Key definitions

Sample Space, S

- Coin flip
  $S = \{\text{Heads}, \text{Tails}\}$
- Flipping two coins
  $S = \{(\text{H,H}), (\text{H,T}), (\text{T,H}), (\text{T,T})\}$
- Roll of 6-sided die
  $S = \{1, 2, 3, 4, 5, 6\}$
- # emails in a day
  $S = \{x \mid x \in \mathbb{Z},\ x \geq 0\}$
- TikTok hours in a day
  $S = \{x \mid x \in \mathbb{R}, 0 \leq x \leq 24\}$

Event, E

- Flip lands heads
  $E = \{\text{Heads}\}$
- $\geq$ 1 head in two coin flips
  $E = \{(\text{H,H}), (\text{H,T}), (\text{T,H})\}$
- Roll is 3 or less:
  $E = \{1, 2, 3\}$
- Low email day ($\leq$ 100 emails)
  $E = \{x \mid x \in \mathbb{Z},\ 0 \leq x \leq 100\}$
- Lost day ($\geq$ 5 TikTok hours):
  $E = \{x \mid x \in \mathbb{R}, 5 \leq x \leq 24\}$

What is a probability?

A number between 0 and 1
to which we ascribe meaning.*

*our belief that an event E occurs.

What is a probability?

this is often referred to as the frequentist's definition of probability

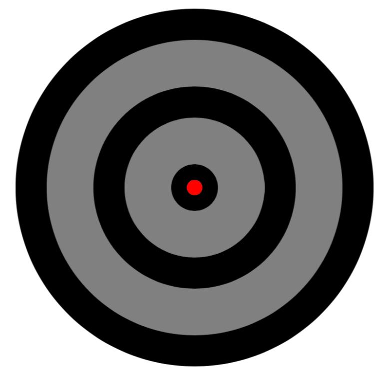
$$P(E) = \lim_{n \to \infty} \frac{n(E)}{n}$$

n = # of total trials

$n(E)$ = # trials where E occurs

Let E = the set of outcomes where you hit the target.

Hit: 0
Thrown: 0



$P(E) \approx$

What is a probability?

$$P(E) = \lim_{n\to\infty} \frac{n(E)}{n}$$

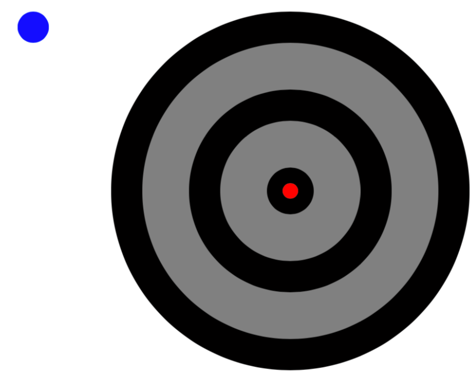
n = # of total trials

$n(E)$ = # trials where E occurs

strawman: if this is the only evidence you see, wouldn't you think, at least for the moment, that hitting the target is impossible?

Let E = the set of outcomes where you hit the target.

```
Hit: 0
Thrown: 1
```



$P(E) \approx 0.00$

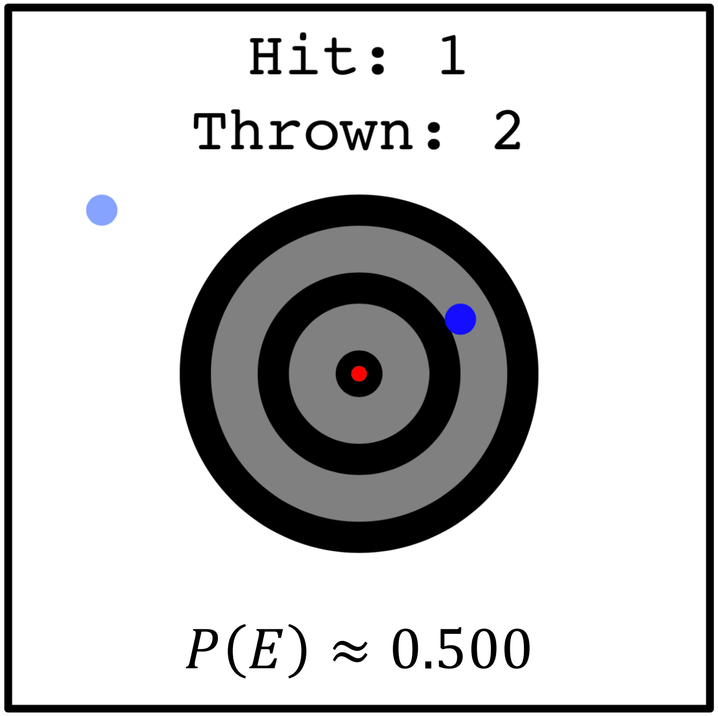
What is a probability?

Let E = the set of outcomes where you hit the target.

$$P(E) = \lim_{n \to \infty} \frac{n(E)}{n}$$

n = # of total trials

$n(E)$ = # trials where E occurs



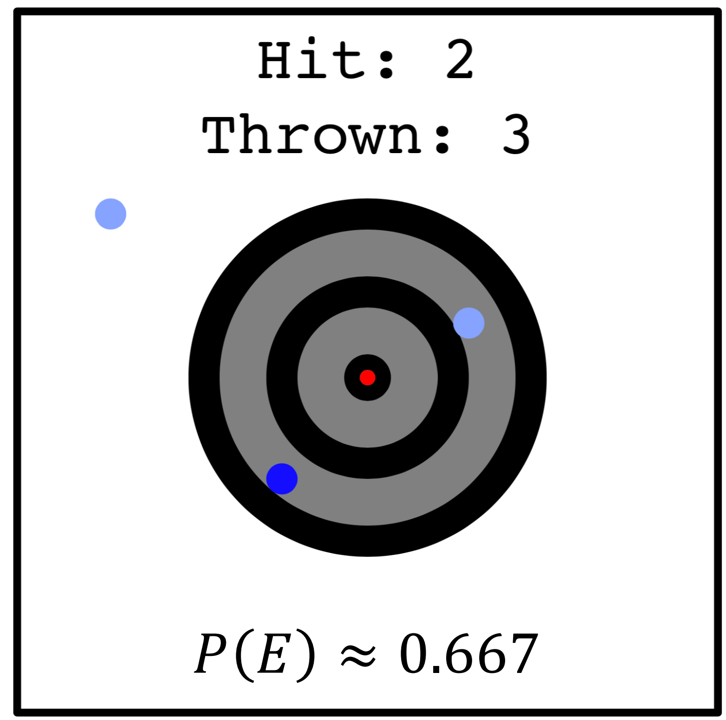
What is a probability?

$$P(E) = \lim_{n\to\infty} \frac{n(E)}{n}$$

n = # of total trials

$n(E)$ = # trials where E occurs

Let E = the set of outcomes where you hit the target.



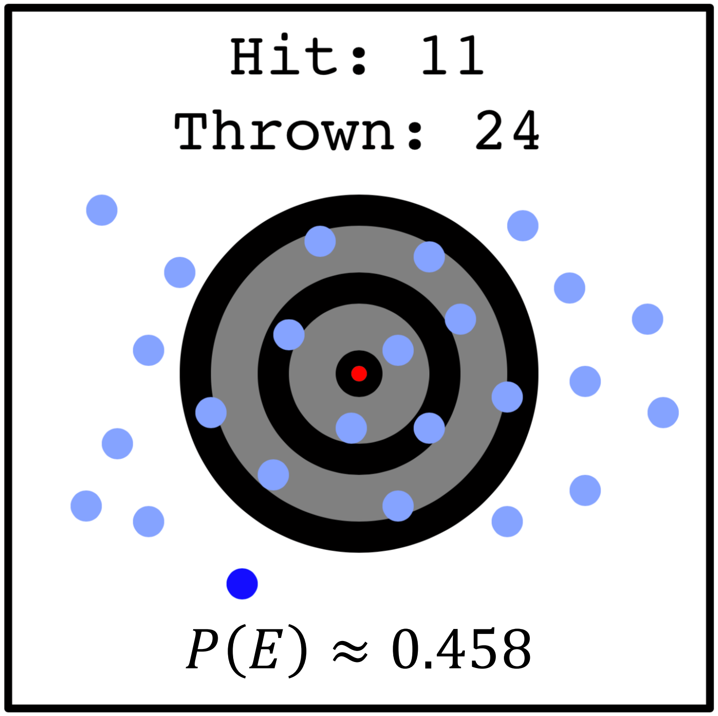
What is a probability?

Let E = the set of outcomes where you hit the target.

$$P(E) = \lim_{n \to \infty} \frac{n(E)}{n}$$

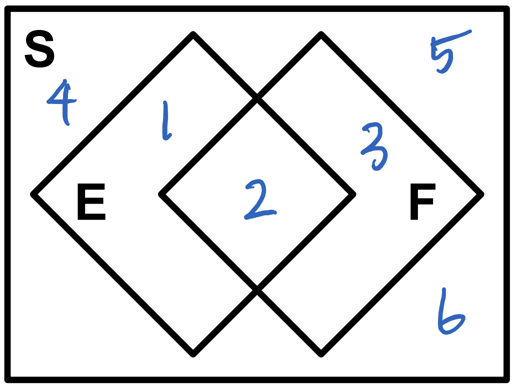
n = # of total trials

$n(E)$ = # trials where E occurs



Axioms of Probability

Quick review of sets



E and F are events in S.

Experiment:

Die roll

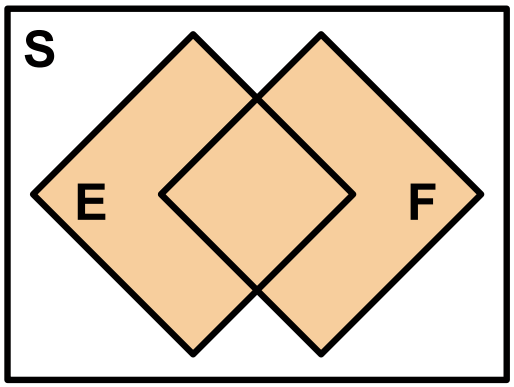
$S = \{1, 2, 3, 4, 5, 6\}$

Let $E = \{1, 2\}$, and $F = \{2,3\}$

2 is in both E and F

4, 5, and 6 are in neither

Quick review of sets



E and F are events in S.
Experiment:
Die roll
$S = \{1, 2, 3, 4, 5, 6\}$
Let $E = \{1, 2\}$, and $F = \{2,3\}$

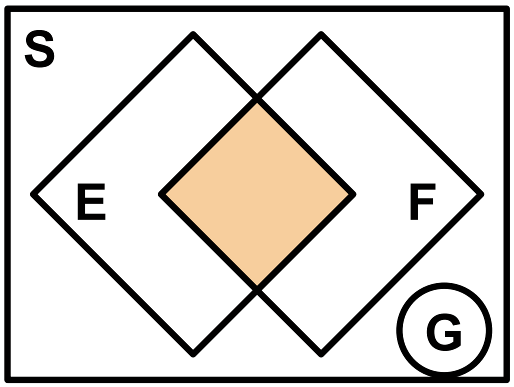
def Union of events, $E \cup F$

The event containing all outcomes in E or F.

$$E \cup F = \{1,2,3\}$$

set theory doesn't ask or even allow us to count 2 twice.
2 is simply present/included.

Quick review of sets



E and F are events in S.
Experiment:
Die roll
$S = \{1, 2, 3, 4, 5, 6\}$
Let $E = \{1, 2\}$, and $F = \{2,3\}$

def Intersection of events, $E \cap F$

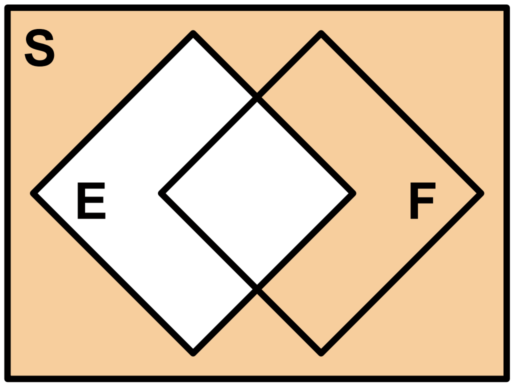
The event containing all outcomes in E and F.

$$E \cap F = EF = \{2\}$$

easier to write this way, so it's written like this more often.

def Mutually exclusive events F and G means that $F \cap G = \emptyset$ perhaps $G = \{6\}$?

Quick review of sets



E and F are events in S.
Experiment:
Die roll
$S = \{1, 2, 3, 4, 5, 6\}$
Let $E = \{1, 2\}$, and $F = \{2,3\}$

<u>def</u> Complement of event E, E^C

The event containing all outcomes in that are ***not*** in E.

$$E^C = \{3, 4, 5, 6\}$$

the complement is everything in the world that isn't in E.

Three Axioms of Probability

Definition of probability: $P(E) = \lim_{n \to \infty} \frac{n(E)}{n}$ } frequentist interpretation of probability

Axiom 1: $0 \leq P(E) \leq 1$

Axiom 2: $P(S) = 1$

Axiom 3: If E and F are mutually exclusive ($E \cap F = \emptyset$), then $P(E \cup F) = P(E) + P(F)$

Axiom 3 is the (analytically) useful axiom

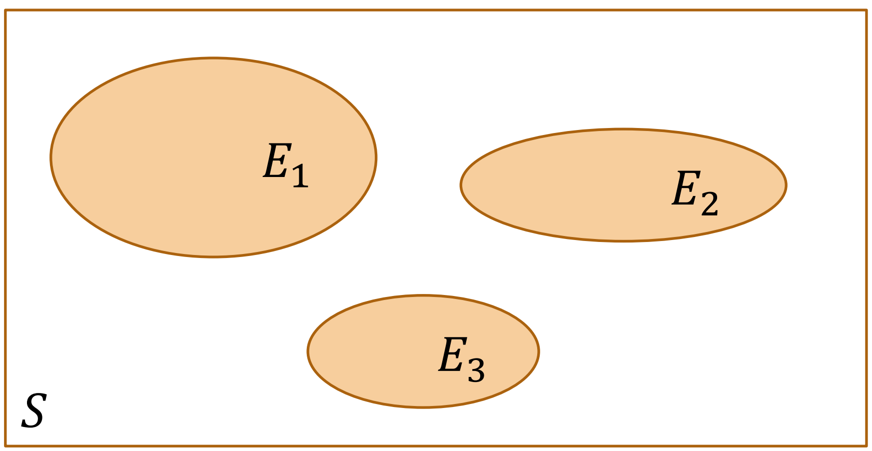
Axiom 3: If E and F are mutually exclusive—that is, if $E \cap F = \emptyset$—then $P(E \cup F) = P(E) + P(F)$

More generally, for any sequence of mutually exclusive events $E_1, E_2, \ldots$:

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

no overlapping events, so no cross terms

represents the grand union of many events



just like the Sum Rule of Counting, but for probabilities

Equally Likely Outcomes

Equally Likely Outcomes

Some sample spaces have **equally likely outcomes**.

- Flipping one coin: S = {Head, Tails}
- Flipping two coins: S = {(H, H), (H, T), (T, H), (T, T)}
- Roll of 6-sided die: S = {1, 2, 3, 4, 5, 6}

If we have equally likely outcomes, then P(Each outcome) $= \dfrac{1}{|S|}$

Therefore $$P(E) = \frac{\#\text{ outcomes in E}}{\#\text{ outcomes in S}} = \frac{|E|}{|S|}$$ (by Axiom 3)

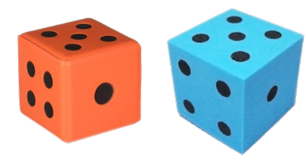
this is the sum of $|E|$ equally likely outcomes, each of which happens with probability $\frac{1}{|S|}$

Roll two dice

assume this, so all outcomes are equally likely

$$P(E) = \frac{|E|}{|S|}$$ Equally likely outcomes

Roll two 6-sided fair dice. What is P(sum = 7)?



$$\begin{aligned} S = \{ & (1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), \\ & (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), \\ & (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), \\ & (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), \\ & (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6), \\ & (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\} \end{aligned}$$

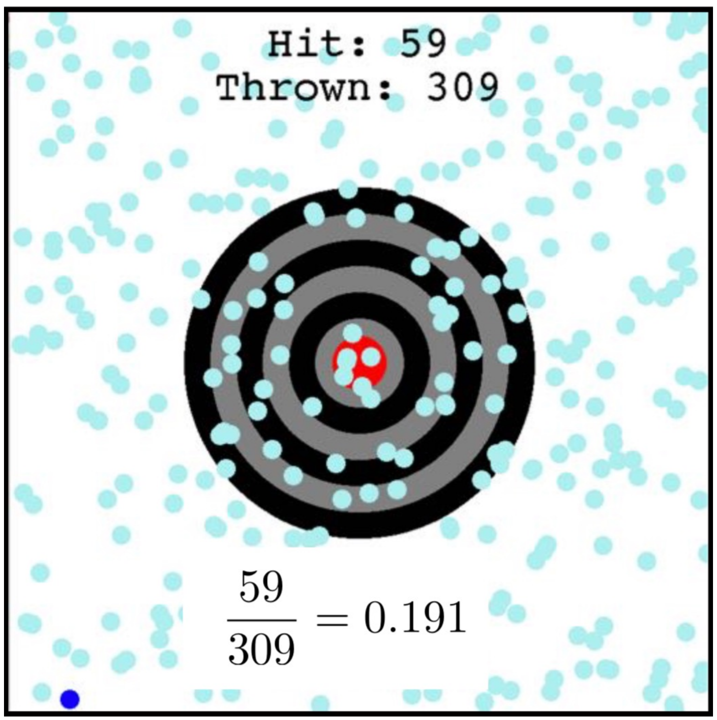
$$E = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$$

$$P(E) = P(E_{(1,6)}) + P(E_{(2,5)}) + \dots + P(E_{(6,1)}) = 6\,\frac{1}{36} = \frac{1}{6}$$

Target revisited

$P(E) = \frac{|E|}{|S|}$ Equally likely outcomes

Let E = the set of outcomes where you hit the target.



Screen size = 800×800

Radius of target: 200

The dart is equally likely to land anywhere on the screen. What is $P(E)$, the probability of hitting the target?

think of each pixel as an equally likely target, and count pixels.

$|S| = 800^2$ $\quad$ $|E| \approx \pi \cdot 200^2$

$$P(E) = \frac{|E|}{|S|} \approx \frac{\pi \cdot 200^2}{800^2} \approx 0.1963$$

Cats and sharks (note: stuffed animals)

$P(E) = \frac{|E|}{|S|}$ Equally likely outcomes

4 cats and 3 sharks in a bag. 3 drawn.

What is P(1 cat and 2 sharks drawn)?

Make indistinct items distinct to get equally likely outcomes.

Question: Do indistinct objects give you an equally likely sample space?

(No)

A. $\frac{3}{7}$

B. $\frac{1}{4} \cdot \frac{2}{3}$

C. $\frac{4}{7} + 2 \cdot \frac{3}{6}$

D. $\frac{12}{35}$

we will derive two different ways.

E. 0

Cats and sharks (ordered solution)

$P(E) = \frac{|E|}{|S|}$ Equally likely outcomes

4 cats and 3 sharks in a bag. 3 drawn.

What is P(1 cat and 2 sharks drawn)?

pretend all stuffed animal are unique

Make indistinct items distinct to get equally likely outcomes.

Define

- S = Pick 3 distinct items, and retain order
- E = 1 distinct cat, 2 distinct sharks

$|S| = 7 \cdot 6 \cdot 5$

$|E|$ = sum of three distinct cases

c stands for cat

s stands for shark

$$\left.\begin{aligned} |E_{css}| &= 4 \cdot 3 \cdot 2 = 24 \\ |E_{scs}| &= 3 \cdot 4 \cdot 2 = 24 \\ |E_{ssc}| &= 3 \cdot 2 \cdot 4 = 24 \end{aligned}\right\} 72$$

$\therefore$ probability is $P(E) = \frac{|E|}{|S|} = \frac{72}{210} = \frac{12}{35}$

Cats and sharks (unordered solution)

$P(E) = \frac{|E|}{|S|}$ Equally likely outcomes

4 cats and 3 sharks in a bag. 3 drawn.

What is P(1 cat and 2 sharks drawn)?

Make indistinct items distinct to get equally likely outcomes.

Define

- S = Pick 3 distinct items, ignore order
- E = 1 distinct cat, 2 distinct sharks

because we're ignoring order with this approach, we rely on combinations and choose terms instead of multiplication

$$|S| = \binom{7}{3} = 35$$

$$|E| = \binom{4}{1} \cdot \binom{3}{2} = 4 \cdot 3 = 12$$

number of ways to choose one cat from four

number of ways to choose any two of three sharks

$$P(E) = \frac{|E|}{|S|} = \frac{12}{35}$$

Exercises

CS109 so far

Review

Counting tasks on n objects

- Sort objects (permutations)
  - Distinct (distinguishable): $n!$
  - Some distinct: $\frac{n!}{n_1!\, n_2! \cdots n_r!}$
- Choose k objects (combinations)
  - Distinct
    - 1 group: $\binom{n}{k}$
    - r groups: $\binom{n}{n_1, n_2, \cdots, n_r}$
- Put objects in r buckets
  - Distinct: r^n
  - Indistinct: $\frac{(n+r-1)!}{n!\,(r-1)!}$

Combinatorics

Equally likely outcomes:

$$P(E) = \frac{|E|}{|S|}$$

Probability

Counting? Probability? Distinctness?

Review

We choose **3 books** from a set of

but distinguishable from the first four

4 distinct (distinguishable) and **2 indistinct** (indistinguishable) books.
Each set of 3 books is equally likely.

Let event $E =$ our choice excludes one or both indistinct books.

1. How many distinct outcomes are in E? restated, how many visibly different subsets

$\binom{4}{2}$ ways to include one of the two copies

$\binom{4}{3}$ ways to exclude both identical copies

$\Rightarrow \binom{4}{2} + \binom{4}{3} = 6 + 4 = 10$

2. What is $P(E)$?

make identical copies distinguishable from each other, else some subsets are more likely than others, and we want equally likely outcomes.

$|E| = 2\binom{4}{2} + \binom{4}{3} = 16$

$|S| = \binom{6}{3} = 20$

$P(E) = \frac{16}{20} = \frac{4}{5}$

distinct, equally likely outcomes

make indistinct → report count

keep distinct → compute probability

Poker Straights and Computer Chips

1. Consider equally likely 5-card poker hands.
   - Define "poker straight" as 5 consecutive rank cards of any suit, suits can vary

   What is P(poker straight)?

- What is an example of an equally likely outcome?
- Should objects be ordered or unordered?

can be either, as long as you're consistent.

2. Computer chips: n chips are manufactured, 1 of which is defective. k chips are randomly selected from n for testing.

   What is P(defective chip is in k selected chips?)

1. Any Poker Straight

assume Ace can be either low or high

possibilities: A2345
23456
34567
⋮
10JQKA

Consider equally likely 5-card poker hands.

- "straight" is 5 consecutive rank cards of any suit

What is P(Poker straight)?

$|S| = \binom{52}{5}$

Define

- S (unordered)

$|E| = \boxed{10} \cdot \left[\underbrace{\binom{4}{1}}\right]^5$

10, because the lowest rank card in the straight can be one of ten different ranks (rest are constrained)

- E (unordered, consistent with S)

$\binom{4}{1}$ is the number of ways to choose a suit for each of the 5 cards

Compute $P(\text{Poker straight}) = \frac{|E|}{|S|} = \frac{10 \cdot 4^5}{\binom{52}{5}} = 0.00394$

2. Chip defect detection

n chips are manufactured, 1 of which is defective.
k chips are randomly selected from n for testing.

What is P(defective chip is in k selected chips?)

$\binom{1}{1}$ is the number of ways we can choose the one defective chip!

$\binom{n-1}{k-1}$ is the number of ways to choose an additional $k-1$ chips from the $n-1$ good ones.

Define

- S (unordered)
- E (unordered, consistent with S)

$|S| = \binom{n}{k} \rightarrow$ all possible subsets of size k

$|E| = \binom{1}{1}\binom{n-1}{k-1} = \binom{n-1}{k-1}$

Compute

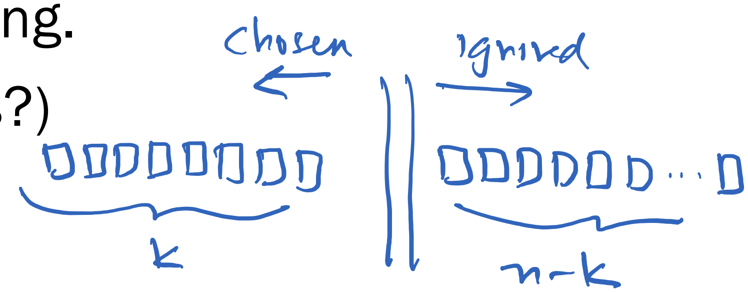
$$P(E) = \frac{|E|}{|S|} = \frac{\binom{n-1}{k-1}}{\binom{n}{k}}$$

$$= \frac{\frac{(n-1)!}{(k-1)!\,\cancel{(n-k)!}}}{\frac{n!}{k!\,\cancel{(n-k)!}}} = \frac{(n-1)!}{n!} \cdot \frac{k!}{(k-1)!} = \frac{k}{n} \Rightarrow \boxed{P(E) = \frac{k}{n}}$$

2. Chip defect detection, solution #2

n chips are manufactured, 1 of which is defective.
k chips are randomly selected from n for testing.

What is P(defective chip is in k selected chips?)



Redefine experiment

1. Choose k indistinct chips (1 way)
2. Throw a dart and make one defective

probability of hitting one to the left of the divider is $\frac{k}{n}$

Define

- S (unordered)
- E (unordered, consistent with S)

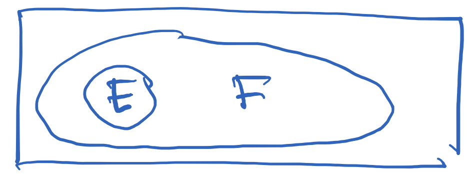
$$\therefore \boxed{P(E) = \frac{k}{n}}$$

Corollaries of Probability

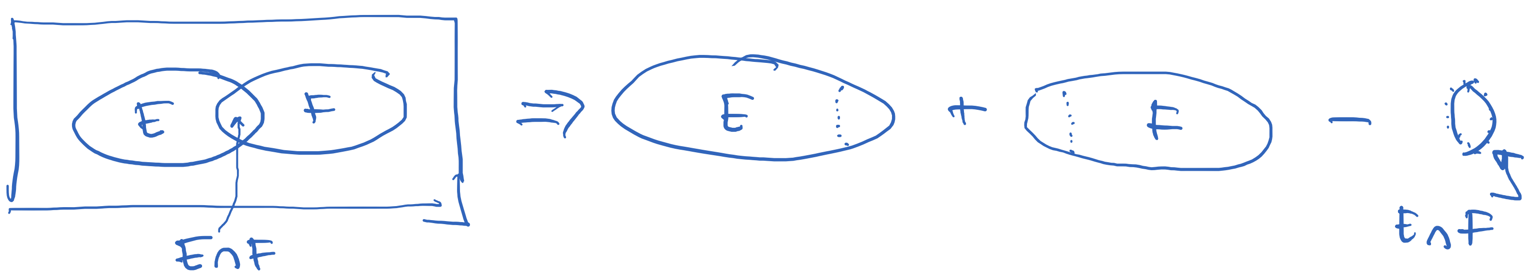
3 Corollaries of Axioms of Probability

Corollary 1: $P(E^C) = 1 - P(E)$

Corollary 2: If $E \subseteq F$, then $P(E) \leq P(F)$



Corollary 3: $P(E \cup F) = P(E) + P(F) - P(EF)$
(Inclusion-Exclusion Principle for Probability)



Selecting Programmers

- P(student programs in Python) = 0.28
- P(student programs in C++) = 0.07
- P(student programs in Python and C++) = 0.05.

What is P(student does not program in (Python or C++))?

1. Define events & state goal
2. Identify known probabilities
3. Solve

Y = student codes in Python
D = student codes in C++

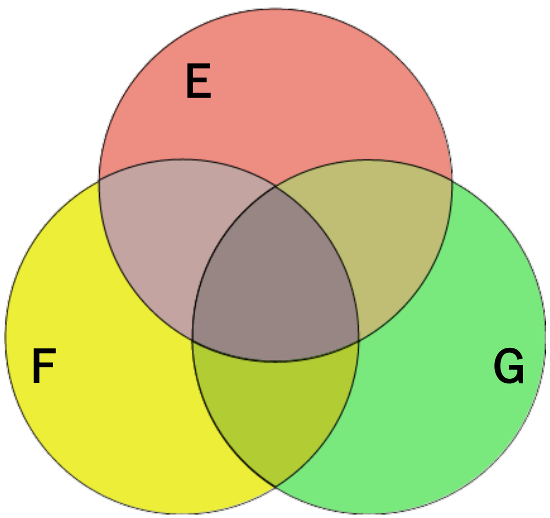
we want

$$P((Y \cup D)^c) = 1 - P(Y \cup D)$$
$$= 1 - (0{,}28 + 0{,}07 - 0{,}05) = 0{,}7$$

Inclusion-Exclusion Principle (Corollary 3)

Corollary 3: $P(E \cup F) = P(E) + P(F) - P(EF)$

General form: $$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{r=1}^{n} (-1)^{(r+1)} \sum_{i_1 < \cdots < i_r} P\left(\bigcap_{j=1}^{r} E_{i_j}\right)$$



$P(E \cup F \cup G) =$

r = 1: $P(E) + P(F) + P(G)$

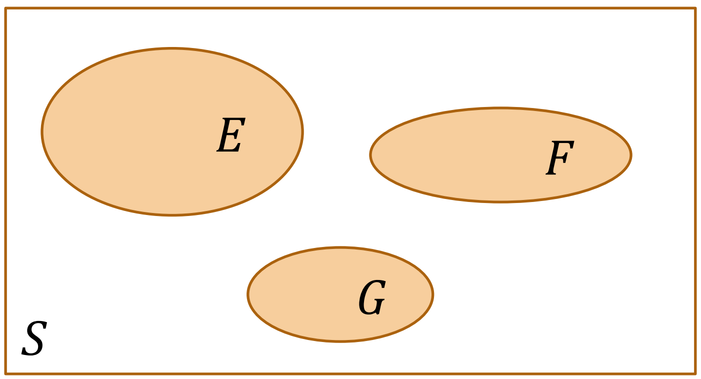
r = 2: $- P(E \cap F) - P(E \cap G) - P(F \cap G)$

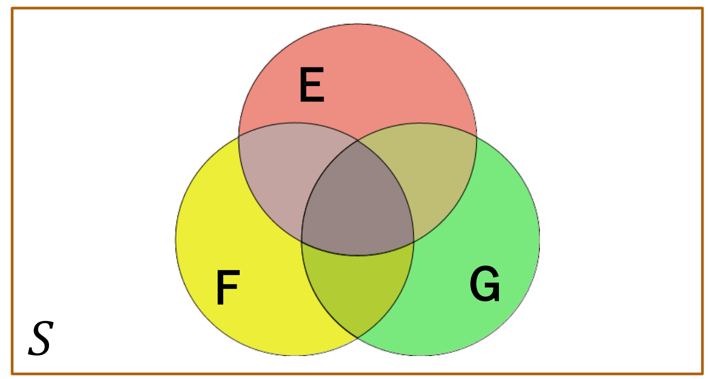
r = 3: $+ P(E \cap F \cap G)$

Takeaway: Union of events

Review

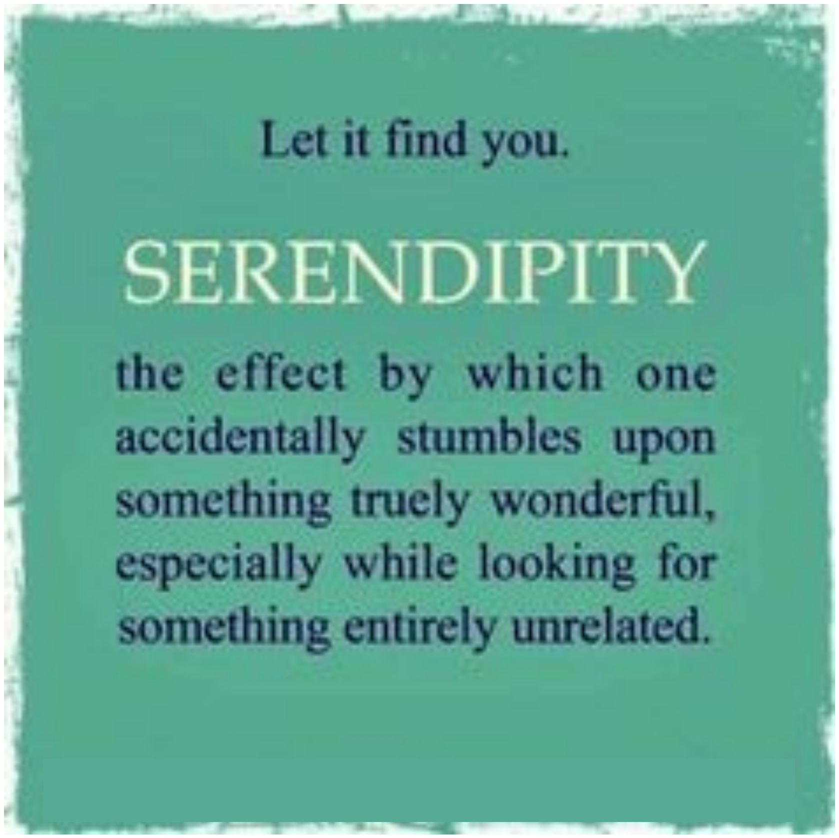
Axiom 3,
Mutually exclusive events $\xrightarrow{\text{generalizes to}}$ Corollary 3,
Inclusion-Exclusion Principle





The challenge of probability is in defining events.
Some event probabilities are easier to compute than others.

Serendipity



Serendipity

- The population of Stanford is $n = 17{,}000$ people.
- You are friends with $r = 100$ people.
- Walk into a room, see $k = 223$ random people.
- Assume each group of k Stanford people is equally likely to be in the room.

What is the probability that you see someone you know in the room?

http://web.stanford.edu/class/cs109/demos/serendipity.html

Serendipity

- The population of Stanford is $n = 17{,}000$ people.
- You are friends with $r = 100$ people.
- Walk into a room, see $k = 223$ random people.
- Assume each group of k Stanford people is equally likely to be in the room.

What is the probability that you see at least one friend in the room?

Define

- S (unordered)
- E: ≥ 1 friend in the room

What strategy would you use?

doable, but tedious

A. $P(\text{exactly } 1) + P(\text{exactly } 2)$
$P(\text{exactly } 3) + \cdots$

easier

B. $1 - P(\text{see no friends})$

Serendipity

you are friends with 100
you are not friends with 16,900

- The population of Stanford is $n = 17{,}000$ people.
- You are friends with $r = 100$ people.
- Walk into a room, see $k = 223$ random people.
- Assume each group of k Stanford people is equally likely to be in the room.

What is the probability that you see at least one friend in the room?

$$|S| = \binom{17000}{223}$$

$$|E^C| = \binom{100}{0}\binom{16900}{223} = \binom{16900}{223}$$

Define

- S (unordered)
- E: ≥ 1 friend in the room

$$P(E) = 1 - P(E^C) = 1 - \frac{\binom{16900}{223}}{\binom{17000}{223}} = 0.7340$$

It is often much easier to compute $P(E^c)$.

The Birthday ~~Paradox~~ Problem

What is the probability that in a set of *n* people, at least one pair of them share the same birthday?

For you to think about (and discuss in your first section)

Card Flipping

In a 52-card deck, cards are flipped one at a time.
After the first ace (of any suit) appears, consider the next card.

Is P(next card = Ace Spades) < P(next card = 2 Clubs)?

Card Flipping

In a 52-card deck, cards are flipped one at a time.
After the first ace (of any suit) appears, consider the next card.

Is P(next card = Ace Spades) < P(next card = 2 Clubs)?

Sample space S = 52 in-order cards (shuffle deck)

Event

E_{AS}, next card is Ace Spades

1. Take out Ace of Spades.
2. Shuffle leftover 51 cards.
3. Add Ace Spades after first ace.

$|E_{AS}| = 51! \cdot 1$

E_{2C}, next card is 2 Clubs

1. Take out 2 Clubs.
2. Shuffle leftover 51 cards.
3. Add 2 Clubs after first ace.

$|E_{2C}| = 51! \cdot 1$

$$P(E_{AS}) = P(E_{2C})$$