05: Independence

Jerry Cain

April 10th, 2024

Lecture Discussion on Ed

Independence I

Independence

Two events E and F are defined as independent if:

$$P(EF) = P(E)P(F)$$

Otherwise E and F are called dependent events.

If E and F are independent, then:

$$P(E|F) = P(E)$$

Intuition through proof

Independent events E and F ⟺ $P(EF) = P(E)P(F)$

Statement:

If E and F are independent, then $P(E|F) = P(E)$.

Proof:

$$P(E|F) = \frac{P(EF)}{P(F)}$$

Definition of conditional probability

$$= \frac{P(E)P(F)}{P(F)}$$

Independence of E and F

$$= P(E)$$

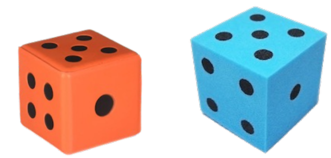
Taking the bus to cancellation city

Knowing that F happened does not change our belief that E happened.

Dice, our misunderstood friends

Independent events E and F $\Leftrightarrow$ $P(EF) = P(E)P(F)$, $P(E|F) = P(E)$

- Roll two 6-sided dice, yielding values D_1 and D_2.
- Let event E: $D_1 = 1$
  event F: $D_2 = 6$
  event G: $D_1 + D_2 = 5$



$G = \{(1,4),(2,3),(3,2),(4,1)\}$

$|G| = 4$

1. Are E and F independent?

$EF = E \cap F = \{(1,6)\}$

$P(E) = 1/6$

$P(F) = 1/6$

$P(EF) = 1/36$

$P(E)P(F) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$

$P(EF) = \frac{1}{36}$

yes!

✅ independent

2. Are E and G independent?

$EG = \{(1,4)\}$

$P(EG) = \frac{1}{36}$

$P(E) = 1/6$

$P(G) = 4/36 = 1/9$

$P(EG) = 1/36 \neq P(E)P(G)$

$1/6 \quad 1/9 \Rightarrow 1/54$

❌ dependent

no!

Generalizing independence

Three events E, F, and G are independent if:

$$\begin{cases} P(EFG) = P(E)P(F)P(G), \text{ and} \\ P(EF) = P(E)P(F), \text{ and} \\ P(EG) = P(E)P(G), \text{ and} \\ P(FG) = P(F)P(G) \end{cases}$$

we need threeway independence

we also need pairwise independence for all possible pairs

n events $E_1, E_2, \ldots, E_n$ are independent if:

for $r = 1, \ldots, n$:

for every subset $E_1, E_2, \ldots, E_r$:

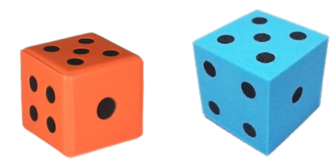
$$P(E_1 E_2 \ldots E_r) = P(E_1)P(E_2) \cdots P(E_r)$$

informally:
- we need pairwise independence for all pairs.
- we need trio-wise independence for all triples.
- we need quartet-wise independence for all quartets.

etc.

Dice, increasingly misunderstood (still our friends)

- Each roll of a 6-sided die is an independent trial.
- Two rolls: D_1 and D_2.
- Let event E: $D_1 = 1$
  event F: $D_2 = 6$
  event G: $D_1 + D_2 = 7$



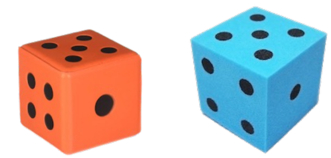
$G = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$

1. Are E and F independent? ✅
   EF is still {(1,6)}
2. Are E and G independent?
3. Are F and G independent?
4. Are E, F, G independent?

$P(EF) = 1/36$

Dice, increasingly misunderstood (still our friends)

- Each roll of a 6-sided die is an independent trial.
- Two rolls: D_1 and D_2.
- Let event E: $D_1 = 1$
  event F: $D_2 = 6$
  event G: $D_1 + D_2 = 7$



$G = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$

1. Are E and F independent? ✅

   $P(EF) = 1/36$

2. Are E and G independent? ✅

   $EG = \{(1,6)\}$
   $P(EG) = 1/36$
   $P(E)P(G) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$

3. Are F and G independent? ✅

   $FG = \{(1,6)\}$
   $P(FG) = 1/36$
   $P(F)P(G) = \frac{1}{6} \cdot \frac{1}{6}$

4. Are E, F, G independent? ❌

   $EFG = \{(1,6)\}$
   $P(EFG) = \frac{1}{36}$
   $P(E)P(F)P(G) = \frac{1}{6^3} \neq \frac{1}{36}$

Pairwise independence is not sufficient to prove independence of 3 or more events!

Independence II

Independent trials

We often are interested in experiments consisting of n **independent trials**.

- n trials, each with the same set of possible outcomes
- n-way independence: an event in one subset of trials is independent of events in other subsets of trials

Examples:

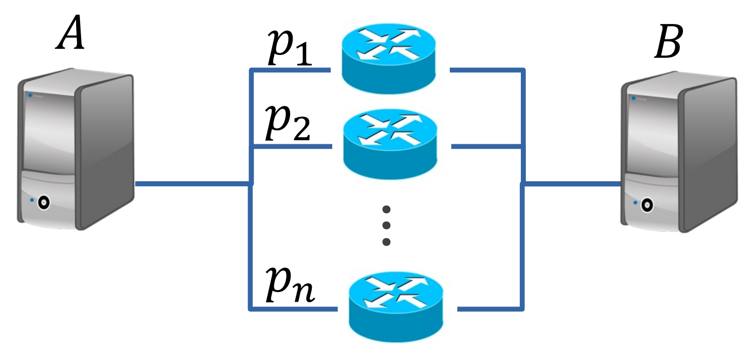
- Flip a coin n times
- Roll a die n times
- Send a multiple-choice survey to n people
- Send n web requests to k different servers

Network reliability

Consider the following parallel network:

- n independent routers, each with probability p_i of functioning (where $1 \leq i \leq n$)
- E = functional path from A to B exists.

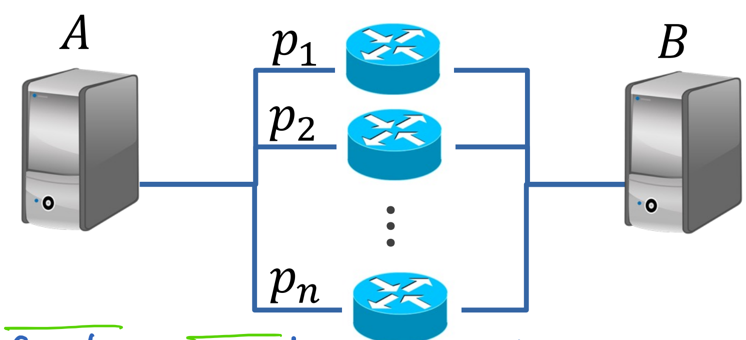
What is $P(E)$?



Network reliability

Consider the following parallel network:

- n independent routers, each with probability p_i of functioning (where $1 \le i \le n$)
- E = functional path from A to B exists.



What is $P(E)$?

router i functions properly with probability p_i
router i fails to function with probability $1-p_i$

$$P(E) = P(\geq 1 \text{ one router works})$$

$$= 1 - P(\text{all routers fail})$$

= 1 − P(router 1 fails AND router 2 fails AND ...)

$$= 1 - (1-p_1)(1-p_2)\cdots(1-p_n)$$

$$= 1 - \prod_{i=1}^{n}(1-p_i)$$

> ≥ 1 with independent trials: take complement

Exercises

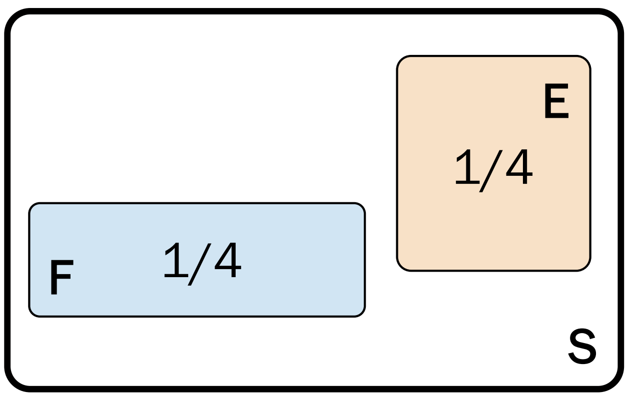
Independence?

Independent events E and F $\Leftrightarrow$ $P(EF) = P(E)P(F)$
$P(E|F) = P(E)$

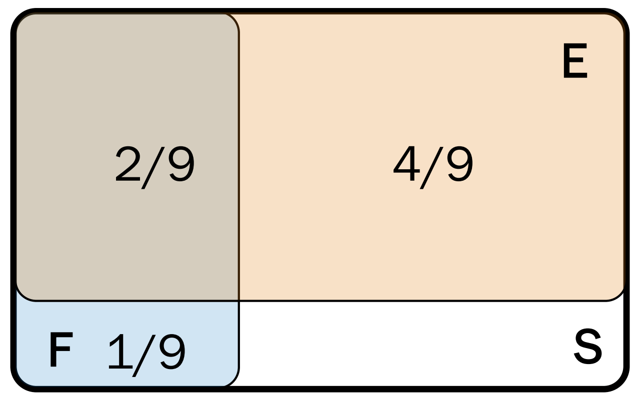
1. True or False? Two events E and F are independent if:
   A. Knowing that F happens means that E can't happen.
   B. Knowing that F happens doesn't change probability that E happened.

2. Are E and F independent in the following pictures?

A.



B.



Independence?

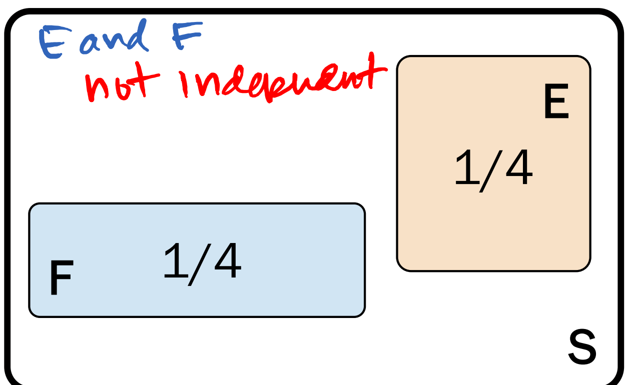
Independent events E and F ⟺ $P(EF) = P(E)P(F)$, $P(E|F) = P(E)$

assume nonzero $P(E), P(F)$

1. True or False? Two events E and F are independent if:
   A. Knowing that F happens means that E can't happen. no! $P(E|F) = 0 \neq P(E)$
   B. Knowing that F happens doesn't change probability that E happened. yes! $P(E|F) = P(E)$ the definition of independence.

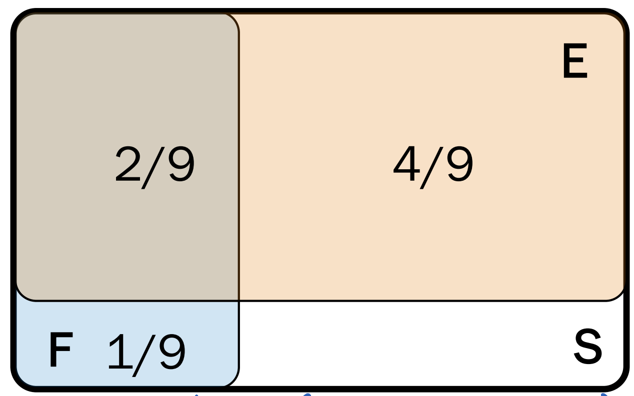
2. Are E and F independent in the following pictures?

A.



$EF = \emptyset$
$P(EF) = 0$
$P(E) = 1/4$
$P(F) = 1/4$
$P(E)P(F) = 1/16 \neq 0$

B.



E and F are independent! :)

$P(E) = \frac{2}{9} + \frac{4}{9} = \frac{6}{9} = \frac{2}{3}$
$P(F) = \frac{2}{9} + \frac{1}{9} = \frac{3}{9} = \frac{1}{3}$
$P(EF) = 2/9$
$P(E)P(F) = \frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9}$

Coin Flips

Suppose we flip a coin n times. Each coin flip is an **independent trial** with probability p of coming up heads. Write an expression for the following:

1. $P(n$ heads on n coin flips)
2. $P(n$ tails on n coin flips)
3. $P($first k heads, then $n-k$ tails)
4. $P($exactly k heads on n coin flips)

Coin Flips

Suppose we flip a coin n times. Each coin flip is an **independent trial** with probability p of coming up heads. Write an expression for the following:

1. $P(n \text{ heads on } n \text{ coin flips})$ → n consecutive heads → HHHH.. H ⇒ p^n
2. $P(n \text{ tails on } n \text{ coin flips})$ → n consecutive tails → TTT … T ⇒ $(1-p)^n = q^n$ where $q = 1-p$
3. $P(\text{first } k \text{ heads, then } n-k \text{ tails})$ → $\underbrace{HH..H}_{k}\underbrace{TTT\ldots T}_{n-k}$ ⇒ $p^k q^{n-k}$
4. $P(\text{exactly } k \text{ heads on } n \text{ coin flips})$
   ↳ any particular sequence of n flips with exactly k heads somewhere within ⇒ $p^k(1-p)^{n-k}$
   and there are $\binom{n}{k}$ such sequences
   total probability is $\binom{n}{k} p^k (1-p)^{n-k}$

$$\binom{n}{k} p^k (1-p)^{n-k}$$

# of mutually exclusive outcomes ($\binom{n}{k}$)

P(a particular outcome's k heads on n coin flips) ($p^k(1-p)^{n-k}$)

Make sure you understand #4! It will come up again.

Probability of events

E or F
$P(E \cup F)$

Mutually exclusive?

Just add!

$P(E) + P(F)$

Inclusion-Exclusion Principle

$P(E) + P(F) - P(E \cap F)$

E and F
$P(EF)$

Independent?

Just multiply!

Chain Rule

Probability of events

E or F
$P(E \cup F)$

Mutually exclusive?

Just add!

$P(E) + P(F)$

Inclusion-Exclusion Principle

$P(E) + P(F) - P(E \cap F)$

E and F
$P(EF)$

Independent?

Just multiply!

$P(E)P(F)$

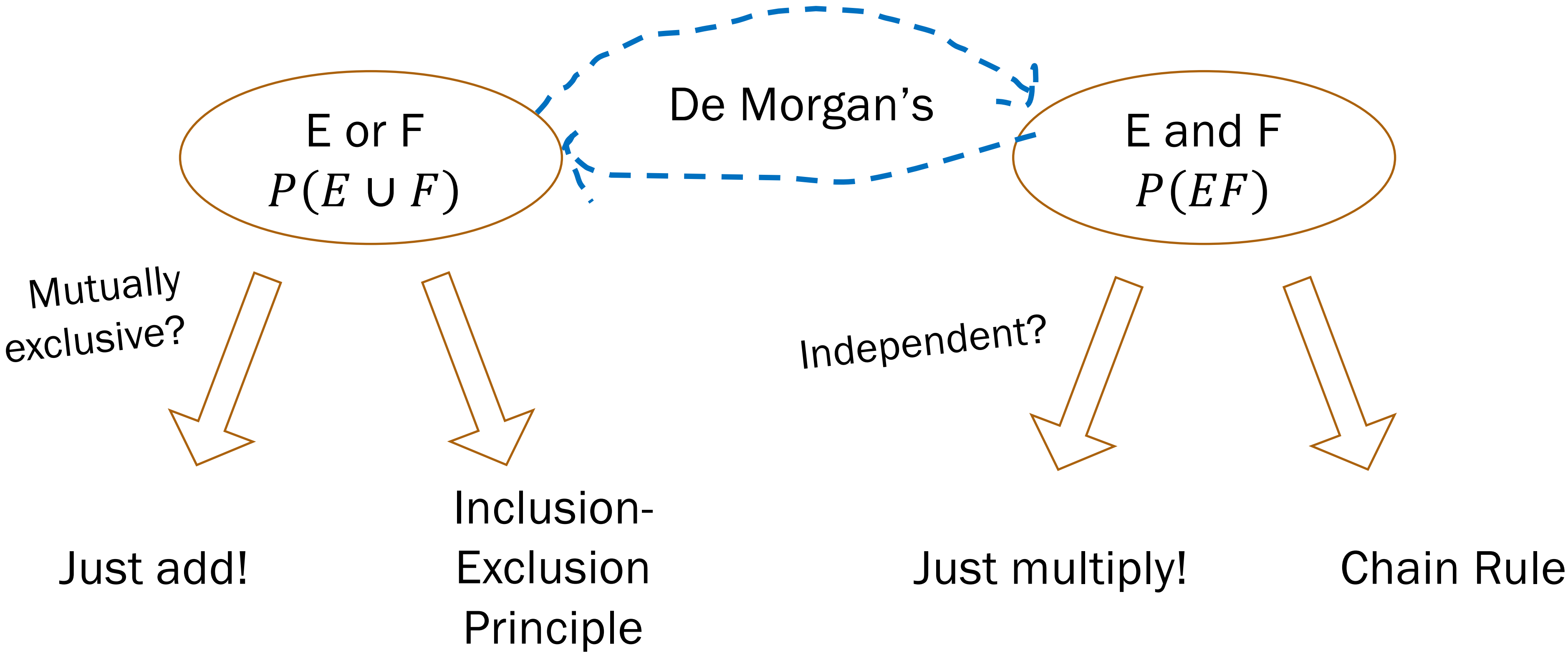
Chain Rule

$P(E)P(F|E)$

or

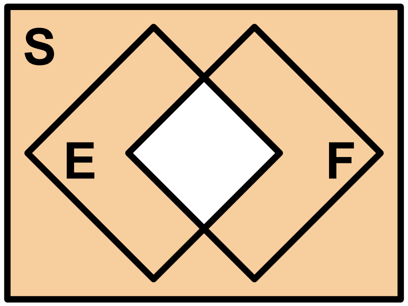
$P(F)P(E|F)$

Probability of events



De Morgan's Laws

De Morgan's lets you switch between AND and OR.



$$(E \cap F)^C = E^C \cup F^C$$

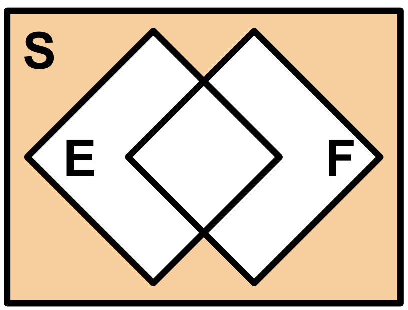
$$\left(\bigcap_{i=1}^{n} E_i\right)^C = \bigcup_{i=1}^{n} E_i^C$$

In probability:

$$P(E_1 E_2 \cdots E_n)$$
$$= 1 - P\left((E_1 E_2 \cdots E_n)^C\right)$$
$$= 1 - P\left(E_1^C \cup E_2^c \cup \cdots \cup E_n^c\right)$$

Great if E_i^C mutually exclusive!

when n=4 $(E_1 \cap E_2 \cap E_3 \cap E_4)^C = E_1^C \cup E_2^C \cup E_3^C \cup E_4^C$



$$(E \cup F)^C = E^C \cap F^C$$

$$\left(\bigcup_{i=1}^{n} E_i\right)^C = \bigcap_{i=1}^{n} E_i^C$$

In probability:

$$P(E_1 \cup E_2 \cup \cdots \cup E_n)$$
$$= 1 - P\left((E_1 \cup E_2 \cup \cdots \cup E_n)^C\right)$$
$$= 1 - P(E_1^c E_2^c \cdots E_n^c)$$

Great if E_i independent!

when n=7

$(E_1 \cup E_2 \cup \cdots \cup E_6 \cup E_7)^C = E_1^C \cap E_2^C \cap \cdots \cap E_6^C \cap E_7^C$

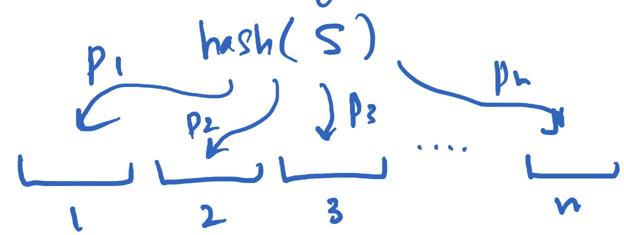
Hash table fun

- m strings are hashed (not uniformly) into a hash table with n buckets. $\sum_{i=1}^{n} p_i = 1$
- Each string hashed is an independent trial w.p. p_i of getting hashed into bucket i.

for each string s, s is hashed to one of n buckets

What is $P(E)$ if

1. E = bucket 1 has $\geq$ 1 string hashed into it?



2. E = at least 1 of buckets 1 to k has $\geq$ 1 string hashed into it?

Hash table fun

- m strings are hashed (not uniformly) into a hash table with n buckets. $\sum_{i=1}^{n} p_i = 1$
- Each string hashed is an independent trial w.p. p_i of getting hashed into bucket i.

What is $P(E)$ if

1. E = bucket 1 has $\geq$ 1 string hashed into it?

Define S_i = string i is hashed into bucket 1
S_i^C = string i is not hashed into bucket 1

$$P(S_i) = p_1$$
$$P(S_i^C) = 1 - p_1$$

Hash table fun

- m strings are hashed (not uniformly) into a hash table with n buckets.
- Each string hashed is an independent trial w.p. p_i of getting hashed into bucket i.

What is $P(E)$ if

1. E = bucket 1 has $\geq$ 1 string hashed into it?

Define S_i = string i is hashed into bucket 1
S_i^C = string i is not hashed into bucket 1

$P(S_i) = p_1$
$P(S_i^C) = 1 - p_1$

WTF (not-real acronym for Want To Find):

$$P(E) = P(S_1 \cup S_2 \cup \cdots \cup S_m)$$

$$= 1 - P\left((S_1 \cup S_2 \cup \cdots \cup S_m)^C\right) \quad \text{Complement}$$

$$= 1 - P\left(S_1^C S_2^C \cdots S_m^C\right) \quad \text{De Morgan's Law}$$

$$= 1 - P\left(S_1^C\right)P\left(S_2^C\right) \cdots P(S_m^C) = 1 - \left(P\left(S_1^C\right)\right)^m \quad S_i \text{ independent trials}$$

$$= 1 - (1 - p_1)^m$$

More hash table fun: Possible approach?

- m strings are hashed (not uniformly) into a hash table with n buckets.
- Each string hashed is an independent trial w.p. p_i of getting hashed into bucket i.

What is $P(E)$ if

1. E = bucket 1 has $\geq$ 1 string hashed into it?
2. E = at least 1 of buckets 1 to k has $\geq$ 1 string hashed into it?

Define F_i = bucket i has at least one string in it

$$
\begin{aligned}
P(E) &= P(F_1 \cup F_2 \cup \cdots \cup F_k) \\
&= 1 - P\left((F_1 \cup F_2 \cup \cdots \cup F_k)^C\right) \\
&= 1 - P\left(F_1^C F_2^C \cdots F_k^C\right) \\
&?= 1 - P\left(F_1^C\right) P\left(F_2^C\right) \cdots P\left(F_k^C\right)
\end{aligned}
$$

⚠ F_i bucket events are *dependent*!

So we cannot approach with complement.

More hash table fun

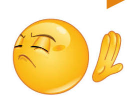
- m strings are hashed (not uniformly) into a hash table with n buckets.
- Each string hashed is an independent trial w.p. p_i of getting hashed into bucket i.

What is $P(E)$ if

1. E = bucket 1 has $\geq$ 1 string hashed into it?
2. E = at least 1 of buckets 1 to k has $\geq$ 1 string hashed into it?

Define F_i = bucket i has at least one string in it

$$
\begin{aligned}
P(E) &= P(F_1 \cup F_2 \cup \cdots \cup F_k) \\
&= 1 - P\left((F_1 \cup F_2 \cup \cdots \cup F_k)^C\right) \\
&= 1 - P\left(F_1^C F_2^C \cdots F_k^C\right)
\end{aligned}
$$



$= P(\text{buckets 1 to } k \textbf{ all denied strings})$

$= \left(P(\text{each string hashes to } k+1 \text{ or higher})\right)^m$

$= (1 - p_1 - p_2 \ldots - p_k)^m$

$$
= 1 - (1 - p_1 - p_2 \ldots - p_k)^m
$$