06: Random Variables

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Lecture Discussion on Ed

Conditional Independence

Conditional Paradigm

For any events A, B, and E, you can condition consistently on E, and all formulas still hold:

when in doubt here, simply assume the E is really just the full sample space of S. then see if the formula reduces to an unconditional one you learned during Week 1 of the quarter.

Axiom 1	$0 \leq P(A\|E) \leq 1$
Corollary 1 (complement)	$P(A\|E) = 1 - P(A^C\|E)$
Transitivity	$P(AB\|E) = P(BA\|E)$
Chain Rule	$P(AB\|E) = P(B\|E)P(A\|BE)$
Bayes' Theorem	$P(A\|BE) = \dfrac{P(B\|AE)P(A\|E)}{P(B\|E)}$

BAE 's theorem?

Conditional Independence

Independent events E and F $\Leftrightarrow$ $P(EF) = P(E)P(F)$, $P(E|F) = P(E)$

Two events A and B are defined as conditionally independent given E if:

$$P(AB|E) = P(A|E)P(B|E)$$

An equivalent definition:

A. $P(A|B) = P(A)$
B. $P(A|BE) = P(A)$
C. $P(A|BE) = P(A|E)$

Conditional Independence

Independent events E and F $\Leftrightarrow$ $P(EF) = P(E)P(F)$
$P(E|F) = P(E)$

Two events A and B are defined as conditionally independent given E if:

$$P(AB|E) = P(A|E)P(B|E)$$

An equivalent definition:

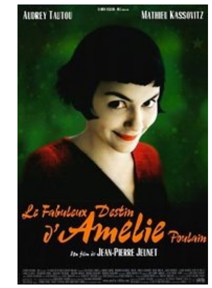
only the third of the three options conditions all probabilities on E.

A. $P(A|B) = P(A)$
B. $P(A|BE) = P(A)$
C. $P(A|BE) = P(A|E)$

E is the "new sample space", so left and right side must both be conditioned on E.

Netflix and Condition

Review



Let E = a user watches Life is Beautiful.

Let F = a user watches Amelie.

What is $P(E)$?

$$P(E) \approx \frac{\text{\# people who have watched movie}}{\text{\# people on Netflix}} = \frac{10{,}234{,}231}{50{,}923{,}123} \approx 0.20$$

What is the probability that a user watches Life is Beautiful, given they watched Amelie?

$$P(E|F) = \frac{P(EF)}{P(F)} = \frac{\text{\# people who have watched both}}{\text{\# people who have watched Amelie}} \approx 0.42$$

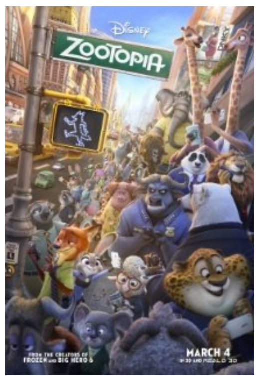
Netflix and Condition

Review

Let E be the event that a user watches the given movie.
Let F be the event that the same user watches Amelie.

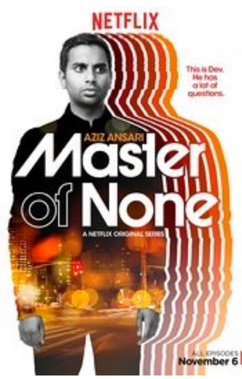
$P(E) = 0.19$

$P(E|F) = 0.14$



$P(E) = 0.32$

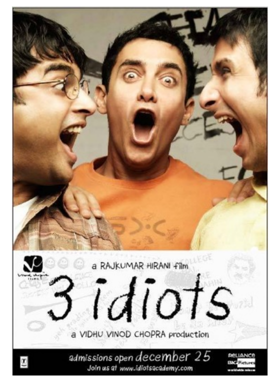
$P(E|F) = 0.35$



$P(E) = 0.20$

$P(E|F) = 0.20$

Independent!



$P(E) = 0.09$

$P(E|F) = 0.72$

$P(E) = 0.20$

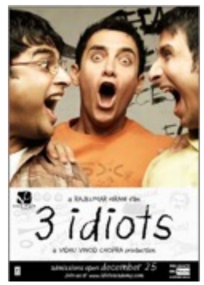
$P(E|F) = 0.42$

Netflix and Condition (on many movies)

Watched:

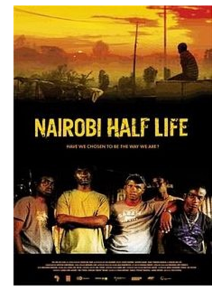
E_1

E_2



E_3

Will they watch?



E_4

What if $E_1 E_2 E_3 E_4$ are not independent? (e.g., all international emotional comedies)

$$P(E_4 | E_1 E_2 E_3) = \frac{P(E_1 E_2 E_3 E_4)}{P(E_1 E_2 E_3)} = \frac{\text{\# people who have watched all 4}}{\text{\# people who have watched those 3}}$$

We need to keep track of an exponential number of movie-watching statistics

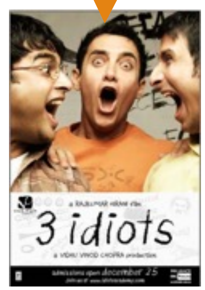
Netflix and Condition (on many movies)

K: likes international emotional comedies

Watched:

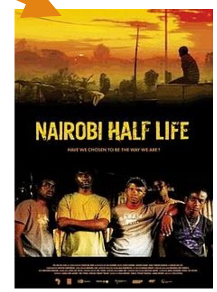
E_1

E_2



E_3

Will they watch?



E_4

Assume: $E_1 E_2 E_3 E_4$ are *conditionally independent given* K

simplifying assumption that work well (or well enough) in practice, even if it's not 100% true.

$$P(E_4|E_1E_2E_3) = \frac{P(E_1E_2E_3E_4)}{P(E_1E_2E_3)}$$

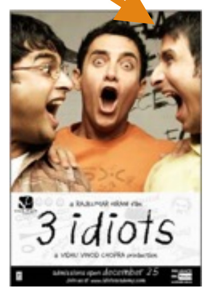
$$P(E_4|E_1E_2E_3K) = \underbrace{P(E_4|K)}_{\text{An easier probability to store and compute!}}$$

Netflix and Condition

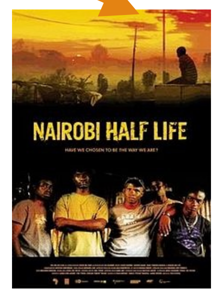
K: likes international emotional comedies

E_1

E_2



E_3



E_4

Challenge: How do we determine K? Stay tuned in 6 weeks' time!

foreshadowing: machine learning

$E_1 E_2 E_3 E_4$ are dependent

$E_1 E_2 E_3 E_4$ are conditionally independent given K

Dependent events can be conditionally independent.
(**And vice versa:** Independent events can be conditionally dependent.)

Random Variables

Random variables are like typed variables

type name value

```
int a = 5;

double b = 4.2;

bit c = 1;
```

CS variables

A is the number of Pokemon we bring to our *future* battle.

$$A \in \{1, 2, \ldots, 6\}$$

B is the amount of money we get *after* we win a battle.

$$B \in \mathbb{R}^+$$

C is 1 *if* we successfully beat the Elite Four. 0 otherwise.

$$C \in \{0,1\}$$

Random variables

Random variables are like typed variables (with uncertainty)

Random Variable

A random variable is a real-valued function defined on a sample space.

Experiment → Outcome → $X = k$

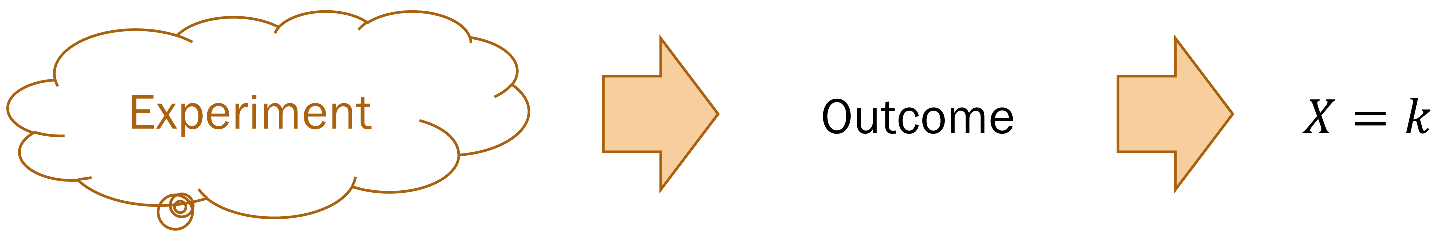
Example:

3 coins are flipped.
Let X = # of heads.
X is a random variable.

1. What is the value of X for the outcomes:
   - (T,T,T)?
   - (H,H,T)?
2. What is the event (set of outcomes) where $X = 2$?
3. What is $P(X = 2)$?

Random Variable

A random variable is a real-valued function defined on a sample space.



Example:

3 coins are flipped.
Let $X = \#$ of heads.
X is a random variable.

1. What is the value of X for the outcomes:
   - (T,T,T)? $X = 0$
   - (H,H,T)? $X = 2$
2. What is the event (set of outcomes) where $X = 2$?
   $\{(H,H,T), (H,T,H), (T,H,H)\}$
   cardinality of this set is 3
3. What is $P(X = 2)$? $P(X=2) = \frac{3}{8}$
   this is really 2^3

Random variables are NOT events!

It is confusing that random variables and events use the same notation.

- **Random variables ≠ events.**
- We can define an event to be a **particular assignment of a random variable, or more generally, in terms of a random variable**.

i.e. $P(X=2)$ or $P(X>0)$

Example:

3 coins are flipped.
Let $X = \#$ of heads.
X is a **random variable**.

$$X = 2$$

event

$$P(X = 2)$$

probability
(number b/t 0 and 1)

Random variables are NOT events!

It is confusing that random variables and events use the same notation.

- **Random variables ≠ events.**
- We can define an event to be a **particular assignment of a random variable, or more generally, in terms of a random variable**.

Example:

3 coins are flipped.
Let X = # of heads.
X is a random variable.

$X = x$	Set of outcomes	$P(X = k)$
$X = \mathbf{0}$	{(T, T, T)}	1/8
$X = \mathbf{1}$	{(H, T, T), (T, H, T), (T, T, H)}	3/8
$X = \mathbf{2}$	{(H, H, T), (H, T, H), (T, H, H)}	3/8 ← just computed this
$X = \mathbf{3}$	{(H, H, H)}	1/8
$X \geq 4$	{ }	0

column adds up to 1

PMF/CDF

So far

3 coins are flipped. Let X = # of heads. X is a random variable.

I went with 2, but it could be any value at all in the support of X.

Experiment → Outcome (flip 2 heads) → $X = \underline{2}$ → $P(X = \underline{2})$

$X = x$	$P(X = k)$	Set of outcomes
$X = \mathbf{0}$	1/8	{(T, T, T)}
$X = \mathbf{1}$	3/8	{(H, T, T), (T, H, T), (T, T, H)}
$X = \mathbf{2}$	3/8	{(H, H, T), (H, T, H), (T, H, H)}
$X = \mathbf{3}$	1/8	{(H, H, H)}
$X \geq 4$	0	{ }

Can we get a "shorthand" for this last step?
Seems like it might be useful!

Probability Mass Function

3 coins are flipped. Let X = # of heads. X is a random variable.

parameter/input k

A function on k
with range [0,1]

$$P(X = k)$$

return value/output
number between
0 and 1

presumably defined
in terms of k.

What would be a *useful* function to define?

The probability of the event that a random variable X takes on the value k!

For **discrete random variables**, this is a **probability mass function**.

Probability Mass Function

3 coins are flipped. Let X = # of heads. X is a random variable.

A function on k
with range [0,1]

2

parameter/input k:
a value of X

$$P(X = 2)$$

0.375

return value/output:
probability of the event
$X = 2$

probability mass function

```
def prob_x(n, k, p):
    n_ways = math.comb(n, k)
    p_way = p ** k * (1 - p) ** (n - k)
    return n_ways * p_way
```

seems like
$P(X=k) = \binom{3}{k} 0.5^3$
for any supported
value of k.

Discrete RVs and Probability Mass Functions

A random variable X is discrete if it can take on countably many values.

- $X = x$, where $x \in \{x_1, x_2, x_3, \dots\}$

The probability mass function (PMF) of a discrete random variable is

$$P(X = x) = \underbrace{p(x) = p_X(x)}_{\text{shorthand notation}}$$

- Probabilities must sum to 1: $\boxed{\sum_{i=1}^{\infty} p(x_i) = 1}$

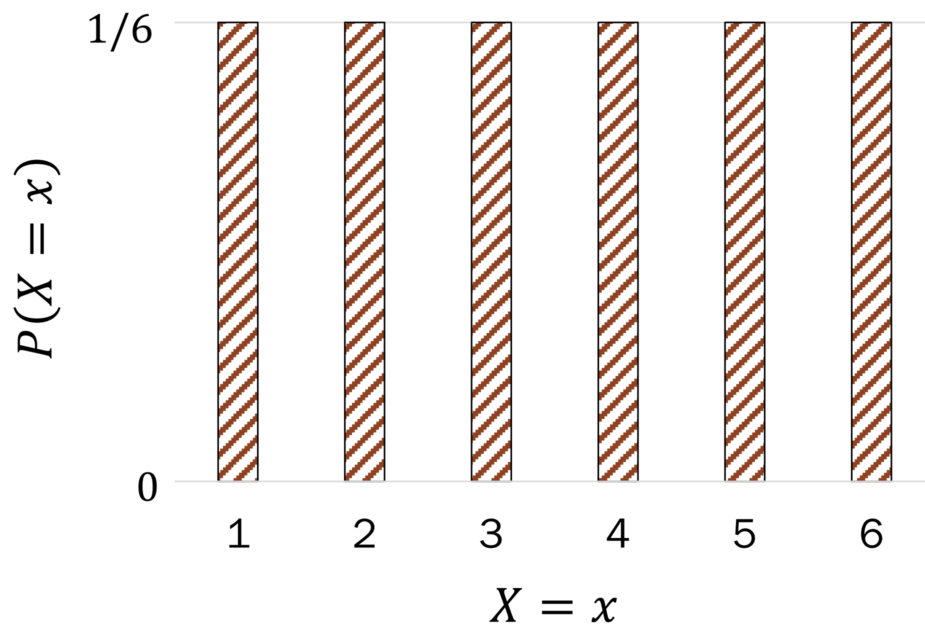
This last point is a good way to verify any PMF you create is valid

PMF for a single 6-sided die

Let X be a random variable that represents the result of a single dice roll.

- Support of X : $\{1, 2, 3, 4, 5, 6\}$
- Therefore, X is a discrete random variable.
- PMF of X:

$$p(x) = \begin{cases} 1/6 & x \in \{1, \dots, 6\} \\ 0 & \text{otherwise} \end{cases}$$



Cumulative Distribution Functions

For a random variable X, the cumulative distribution function (CDF) is defined as

$$F(a) = F_X(a) = P(X \leq a), \text{where } -\infty < a < \infty$$

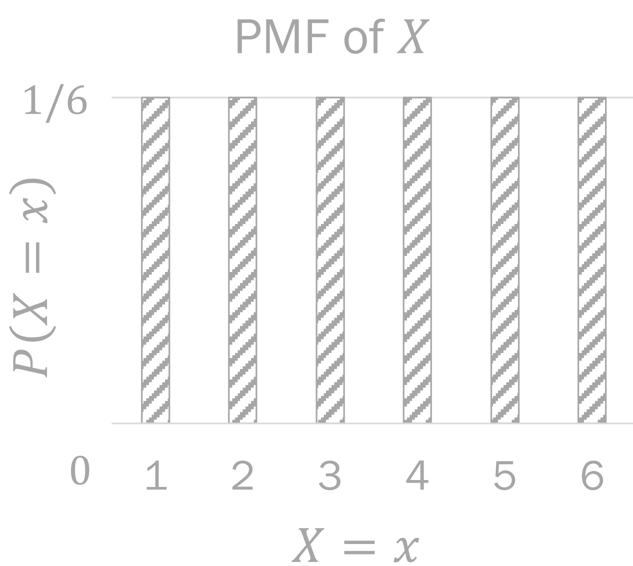
For a discrete RV X, the CDF is:

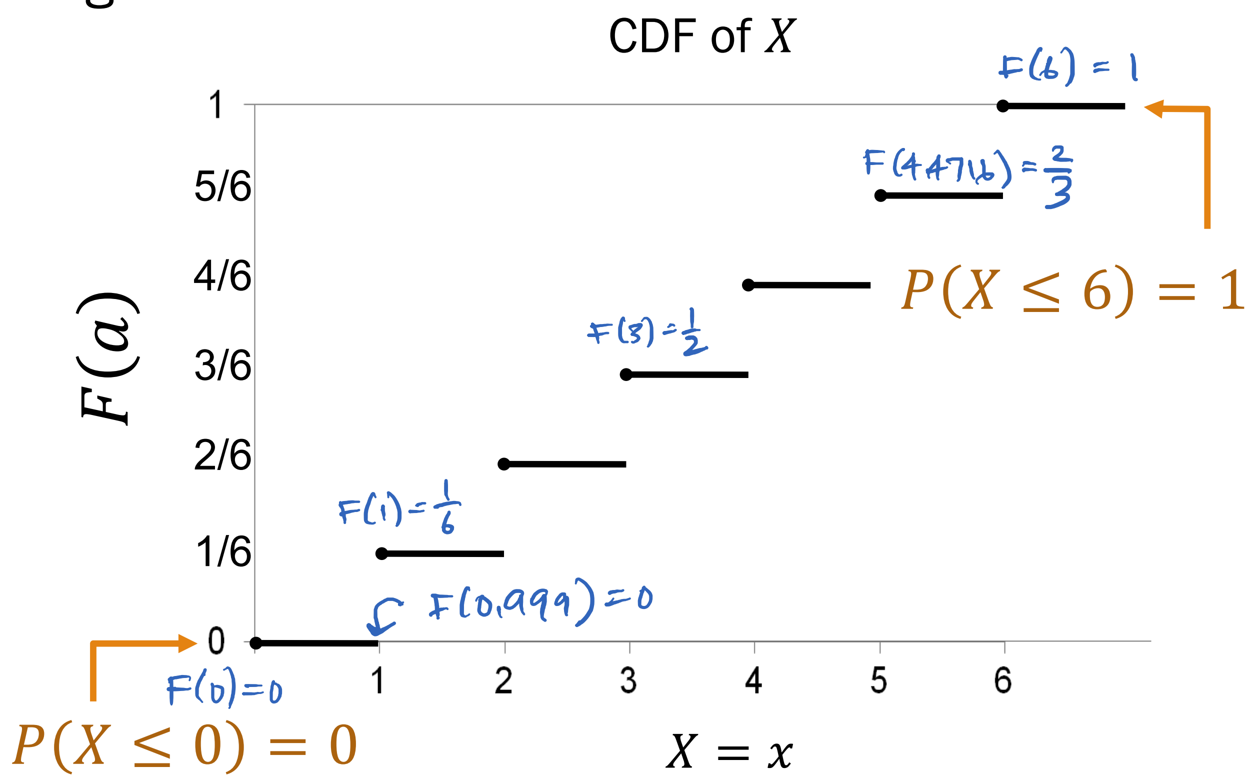
$$F(a) = P(X \leq a) = \sum_{\text{all } x \leq a} p(x)$$

CDFs as graphs

CDF (cumulative distribution function) $F(a) = P(X \leq a)$

Let X be a random variable that represents the result of a single dice roll.





Expectation

Discrete random variables

Experiment outcomes

PMF
$P(X = x) = p(x)$

Definition

Discrete Random Variable, X

CDF $F(x)$

Properties

Support

Without performing the experiment:

- The support tells us which values our random variable might produce
- Next up: How do we report the "average" value?

Expectation

The expectation of a discrete random variable X is defined as:

$$E[X] = \sum_{x:p(x)>0} p(x) \cdot x$$

- Note: sum over all values of $X = x$ that have non-zero probability.
- Other names: **mean**, expected value, **weighted average**,
  center of mass, first moment

these are all used interchangeably, though I prefer weighted average.

Expectation of a die roll

$$E[X] = \sum_{x:p(x)>0} p(x) \cdot x$$ Expectation of X

What is the expected value of a 6-sided die roll?

1. Define random variables

X = RV for value of roll

$$P(X = x) = \begin{cases} 1/6 & x \in \{1, \dots, 6\} \\ 0 & \text{otherwise} \end{cases}$$

2. Solve

$$E[X] = 1\left(\frac{1}{6}\right) + 2\left(\frac{1}{6}\right) + 3\left(\frac{1}{6}\right) + 4\left(\frac{1}{6}\right) + 5\left(\frac{1}{6}\right) + 6\left(\frac{1}{6}\right) = \frac{7}{2}$$

Important properties of expectation

1. Linearity:

$$E[aX + b] = aE[X] + b$$

- Let X = 6-sided dice roll, $Y = 2X - 1$.
- $E[X] = 3.5$
- $E[Y] = 6$

2. Expectation of a sum = sum of expectation:

$$E[X + Y] = E[X] + E[Y]$$

Sum of two dice rolls:

- Let X = roll of die 1, Y = roll of die 2
- $E[X + Y] = 3.5 + 3.5 = 7$

3. Unconscious statistician:

$$E[g(X)] = \sum_x g(x)p(x)$$

These properties let you avoid defining difficult PMFs.

Linearity of Expectation proof

$$E[X] = \sum_{x:p(x)>0} p(x) \cdot x$$

$$E[aX + b] = aE[X] + b$$

Proof:

$$E[aX + b] = \sum_x (ax + b)p(x) = \sum_x axp(x) + bp(x)$$

$$= a \sum_x xp(x) + b \sum_x p(x)$$

$$= a\, E[X] + b \cdot 1$$

Expectation of Sum intuition

$$E[X] = \sum_{x:p(x)>0} p(x) \cdot x$$

$$E[X+Y] = E[X] + E[Y]$$

we'll prove this in a few lectures

the proof technically requires material we've not covered yet.

Intuition for now:

X	Y	$X+Y$
3	6	9
2	4	6
6	12	18
10	20	30
-1	-2	-3
0	0	0
8	16	24

Average:

$$\frac{1}{n}\sum_{i=1}^{n} x_i \quad + \quad \frac{1}{n}\sum_{i=1}^{n} y_i \quad = \quad \frac{1}{n}\sum_{i=1}^{n} (x_i + y_i)$$

$$\frac{1}{7}(28) \quad + \quad \frac{1}{7}(56) \quad = \quad \frac{1}{7}(84)$$

LOTUS proof

$E[g(X)] = \sum_x g(x)p(x)$ Expectation of $g(X)$

Let $Y = g(X)$, where g is a real-valued function.

$$
\begin{aligned}
E[g(X)] = E[Y] &= \sum_j y_j p(y_j) \\
&= \sum_j y_j \sum_{i:g(x_i)=y_j} p(x_i) \\
&= \sum_j \sum_{i:g(x_i)=y_j} y_j \, p(x_i) \\
&= \sum_j \sum_{i:g(x_i)=y_j} g(x_i) \, p(x_i) \\
&= \sum_i g(x_i) \, p(x_i)
\end{aligned}
$$

For you to review so that you can sleep tonight!

Exercises

A Whole New World with Random Variables

Event-driven probability

- Relate only binary events
  - Either something happens (E)
  - or it doesn't happen (E^C)
- Can only report probability
- Lots of combinatorics

Random Variables

- Link multiple similar events together ($X = 1,\ X = 2, \dots, X = 6$)
- Can compute statistics: report the "average" outcome
- Once we have the PMF (for discrete RVs), we can do regular math

Example random variable

Consider 5 flips of a coin which comes up heads with probability p.
Each coin flip is an independent trial. **Let Y = # of heads on 5 flips.**

1. What is the support of Y? In other words, what are the values that Y can take on with non-zero probability?

2. Define the event $Y = 2$. What is $P(Y = 2)$?

3. What is the PMF of Y? In other words, what is $P(Y = k)$, for k in the support of Y?

Example random variable

Consider 5 flips of a coin which comes up heads with probability p. Each coin flip is an independent trial. **Let Y = # of heads on 5 flips.**

1. What is the support of Y? In other words, what are the values that Y can take on with non-zero probability?

   $\{0, 1, 2, 3, 4, 5\}$

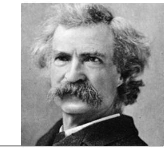
2. Define the event $Y = 2$. What is $P(Y = 2)$?

   $$P(Y = 2) = \binom{5}{2} p^2 (1-p)^3$$

3. What is the PMF of Y? In other words, what is $P(Y = k)$, for k in the support of Y?

   $$P(Y = k) = \binom{5}{k} p^k (1-p)^{5-k}$$

Lying with statistics



A school has 3 classes with 5, 10, and 150 students.
What is the average class size?

1. Interpretation #1

- Randomly choose a <u>class</u> with equal probability.
- X = size of chosen class

$$E[X] = 5\left(\frac{1}{3}\right) + 10\left(\frac{1}{3}\right) + 150\left(\frac{1}{3}\right)$$

$$= \frac{165}{3} = 55$$

What alumni relations usually reports

2. Interpretation #2

- Randomly choose a <u>student</u> with equal probability.
- Y = size of chosen class

$$E[Y] = 5\left(\frac{5}{165}\right) + 10\left(\frac{10}{165}\right) + 150\left(\frac{150}{165}\right)$$

$$= \frac{22635}{165} \approx 137$$

Average student perception of class size

Being a statistician unconsciously

$$E[g(X)] = \sum_x g(x)p(x)$$ Expectation of $g(X)$

Let X be a discrete random variable.

- $P(X = x) = \frac{1}{3}$ for $x \in \{-1, 0, 1\}$

Let $Y = |X|$. What is $E[Y]$?

A. $\frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot -1 \quad = 0$

B. $E[Y] = E[0] \quad = 0$

C. $\frac{1}{3} \cdot 0 + \frac{2}{3} \cdot 1 \quad = \frac{2}{3}$

D. $\frac{1}{3} \cdot |-1| + \frac{1}{3} \cdot |0| + \frac{1}{3}|1| = \frac{2}{3}$

E. C and D

Being a statistician unconsciously

$E[g(X)] = \sum_x g(x)p(x)$ Expectation of $g(X)$

Let X be a discrete random variable.

- $P(X = x) = \frac{1}{3}$ for $x \in \{-1, 0, 1\}$

Let $Y = |X|$. What is $E[Y]$?

A. $\frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot -1 \quad = 0$ ✗ $E[X]$

B. $E[Y] = E[0] \quad = 0$ ✗ $E[E[X]]$

C. $\frac{1}{3} \cdot 0 + \frac{2}{3} \cdot 1 \quad = \frac{2}{3}$

1. Find PMF of Y: $p_Y(0) = \frac{1}{3}, p_Y(1) = \frac{2}{3}$
2. Compute $E[Y]$

D. $\frac{1}{3} \cdot |-1| + \frac{1}{3} \cdot |0| + \frac{1}{3}|1| = \frac{2}{3}$

Use LOTUS by using PMF of X:

1. $P(X = x) \cdot |x|$
2. Sum up

E. C and D