09: Continuous RVs

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Lecture Discussion on Ed

Continuous RVs

People heights

You are volunteering at the local elementary school fundraiser.

- To buy a t-shirt for your friend Vanessa, you need to know her height.

1. What is the probability that your friend is 54.0923857234 inches tall?

   Essentially 0

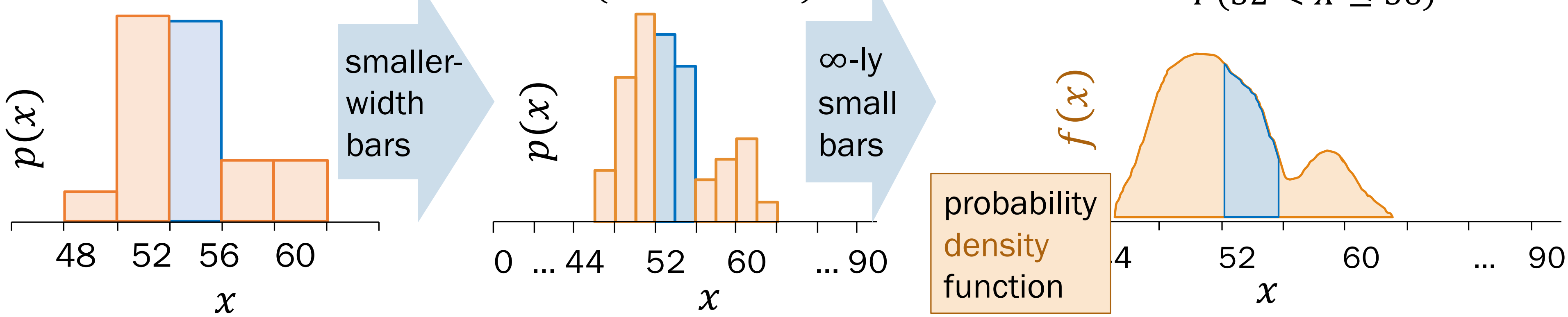
2. What is the probability that Vanessa is between 52–56 inches tall?

stated this way, the probability is nonzero.

$P(52 < X \leq 56)$

$P(52 < X \leq 54) + P(54 < X \leq 56)$

$P(52 < X \leq 56)$



Continuous RV definition

A random variable X is continuous if there is a **probability density function** $f(x) \geq 0$ such that for $-\infty < x < \infty$:

yes, this is an integral

$$P(a \leq X \leq b) = \int_a^b f(x)\,dx$$

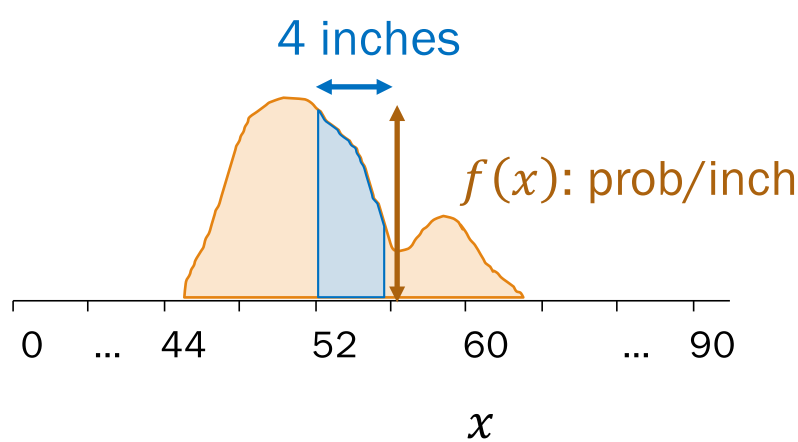
Integrating a PDF must always yield a valid probability, no matter the values of a and b. The PDF must also satisfy:

$$\int_{-\infty}^{\infty} f(x)\,dx = P(-\infty < X < \infty) = 1$$

Note: $f(x)$ is sometimes written as $f_X(x)$ to be clear the random variable is X.

Main takeaway

Integrate $f(x)$ to get probabilities.



PDF Units: probability per units of X

$$P(52 \leq X \leq 56) = \int_{52}^{56} f(x)\, dx$$

PMF vs PDF

Discrete random variable X

Probability mass function (PMF):

$$p(x)$$

To get probability:

$$P(X = x) = p(x)$$

Continuous random variable X

Probability density function (PDF):

$$f(x)$$

To get probability:

$$P(a \leq X \leq b) = \int_a^b f(x)dx$$

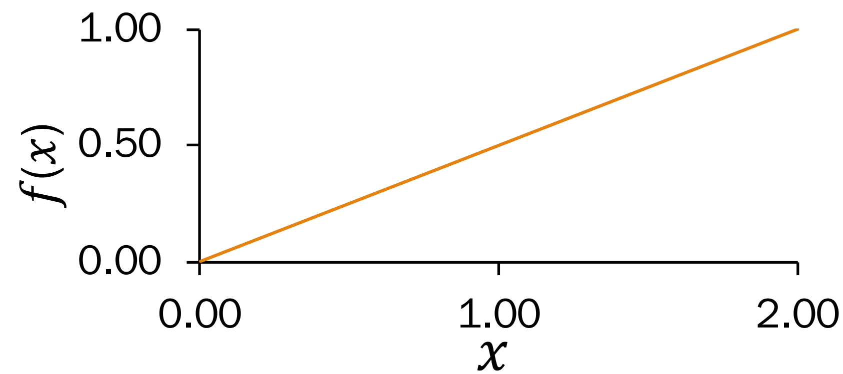
Both are measures of how likely X is to take on a value or some range of values.

Computing probability

$$P(a \leq X \leq b) = \int_a^b f(x)\,dx$$

Let X be a continuous RV with PDF:

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$



What is $P(X \geq 1)$?

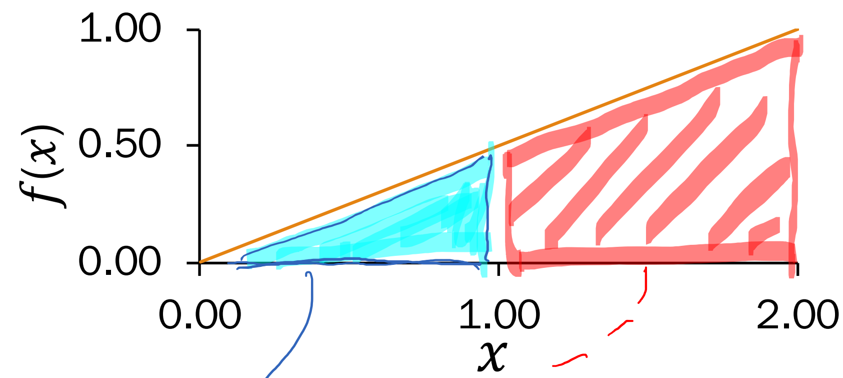
Computing probability

$$P(a \le X \le b) = \int_a^b f(x)\,dx$$

Let X be a continuous RV with PDF:

confirm: $\int_0^2 \frac{x}{2}dx = 1$, so valid pdf

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$



What is $P(X \ge 1)$?

Strategy 1: Integrate

$$P(1 \le X < \infty) = \int_1^\infty f(x)dx = \int_1^2 \frac{1}{2}x\,dx$$

$$= \frac{1}{2}\left(\frac{1}{2}x^2\right)\Big|_1^2 = \frac{1}{2}\left[2 - \frac{1}{2}\right] = \frac{3}{4}$$

Strategy 2: Know triangles

$$1 - \frac{1}{2}\left(\frac{1}{2}\right) = \frac{3}{4}$$

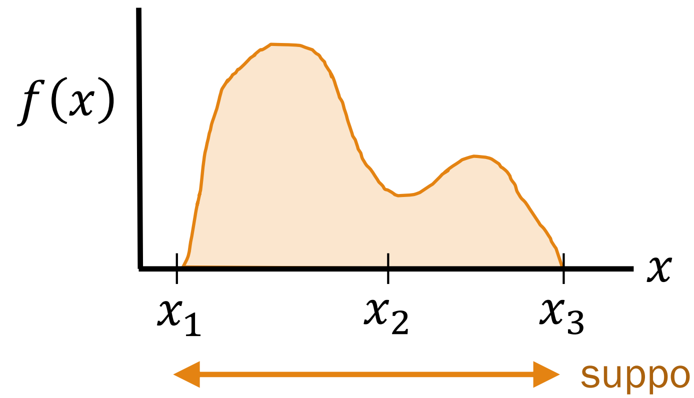
Wait! Is this even legal?

$$P(0 \le X < 1) = \int_0^1 f(x)dx\,??$$

PDF Properties

For a continuous RV X with PDF f,

$$P(a \le X \le b) = \int_a^b f(x)\,dx$$



support: values of x where $f(x) > 0$

True/False:

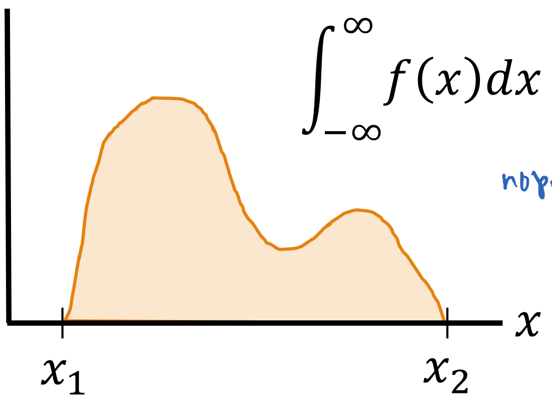
1. $P(X = c) = 0$ — Interval width $dx \to 0$
2. $P(a \le X \le b) = P(a < X < b) = P(a \le X < b) = P(a < X \le b)$
3. $f(x)$ is a probability — It's a probability density!
4. In the graphed PDF above,
   $P(x_1 \le X \le x_2) > P(x_2 \le X \le x_3)$ — Compare area under the curve

Determining valid PDFs

$$P(a \le X \le b) = \int_a^b f(x)\,dx$$

Which of the following functions are valid PDFs?

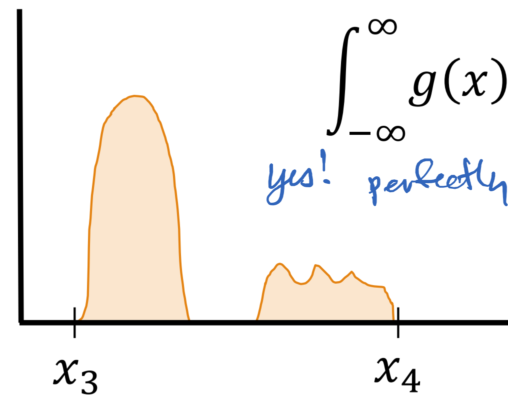
1. $f(x)$



$$\int_{-\infty}^{\infty} f(x)dx = 0.5$$

nope! this integral needs to be 1.0, not 0.5

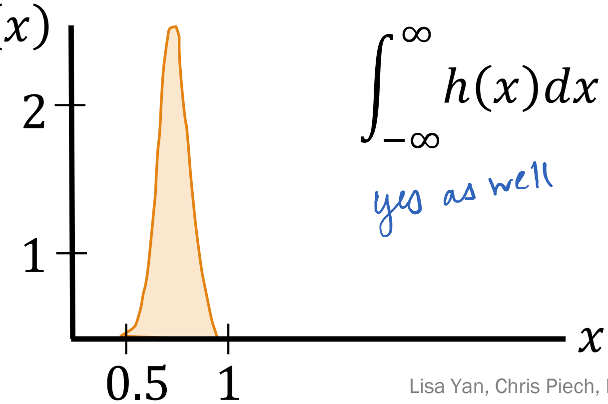
2. $g(x)$



$$\int_{-\infty}^{\infty} g(x)dx = 1$$

yes! perfectly fine

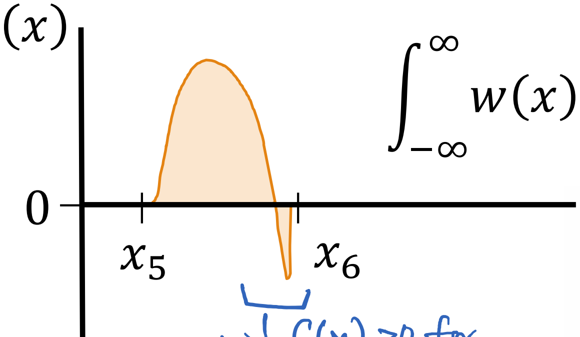
3. $h(x)$



$$\int_{-\infty}^{\infty} h(x)dx = 1$$

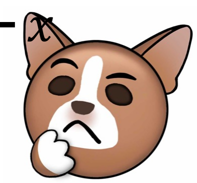
yes as well

4. $w(x)$



$$\int_{-\infty}^{\infty} w(x)dx = 1$$

no! $f(x) > 0$ for valid pdf's



Uniform RV

Uniform Random Variable

<u>def</u> A **Uniform** random variable X is defined as follows:

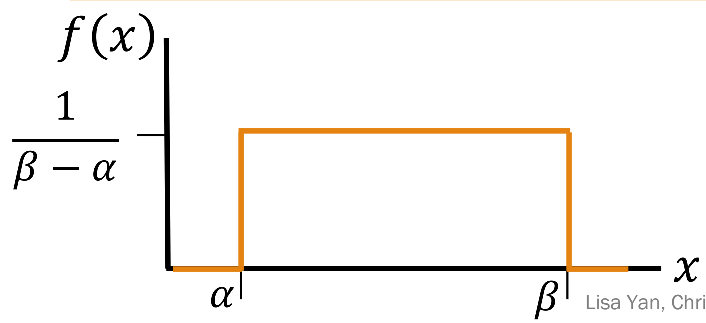
$$X \sim \text{Uni}(\alpha, \beta)$$

Support: $[\alpha, \beta]$
(sometimes defined over (α, β))

PDF $$f(x) = \begin{cases} \frac{1}{\beta - \alpha} & \text{if } \alpha \le x \le \beta \\ 0 & \text{otherwise} \end{cases}$$

Expectation $$E[X] = \frac{\alpha + \beta}{2}$$

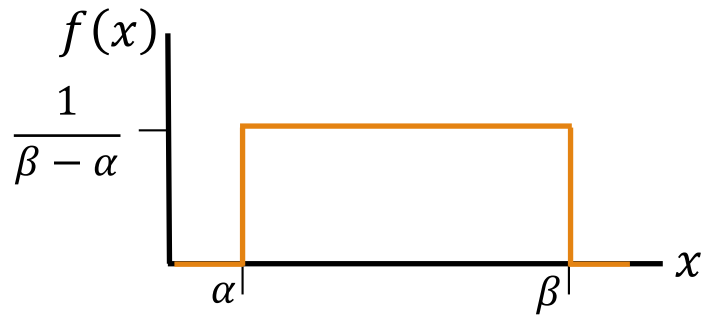
Variance $$\text{Var}(X) = \frac{(\beta - \alpha)^2}{12}$$



Quick check

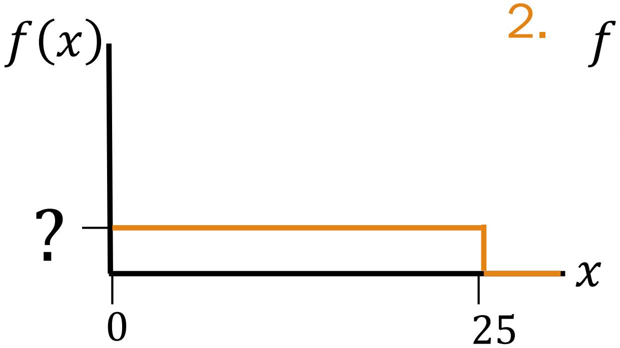
If $X \sim \text{Uni}(\alpha, \beta)$, the PDF of X is:

$$f(x) = \begin{cases} \frac{1}{\beta - \alpha} & \text{if } \alpha \le x \le \beta \\ 0 & \text{otherwise} \end{cases}$$

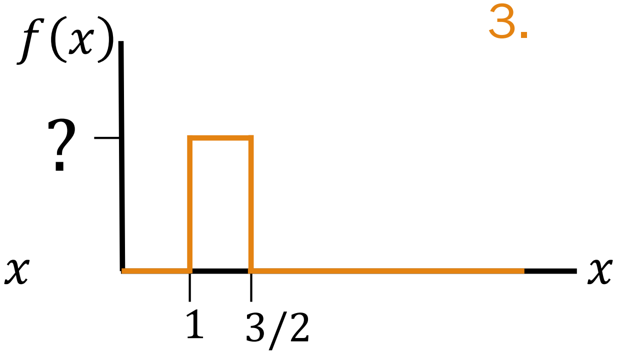


What is $\frac{1}{\beta - \alpha}$ if the following graphs are PDFs of Uniform RVs X?

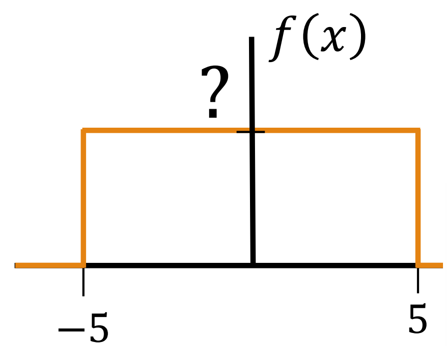
1.



2.



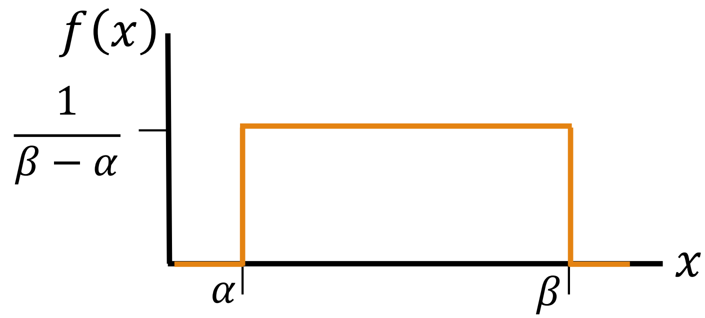
3.



Quick check

If $X \sim \text{Uni}(\alpha, \beta)$, the PDF of X is:

$$f(x) = \begin{cases} \frac{1}{\beta-\alpha} & \text{if } \alpha \le x \le \beta \\ 0 & \text{otherwise} \end{cases}$$



What is $\frac{1}{\beta-\alpha}$ if the following graphs are PDFs of Uniform RVs X?

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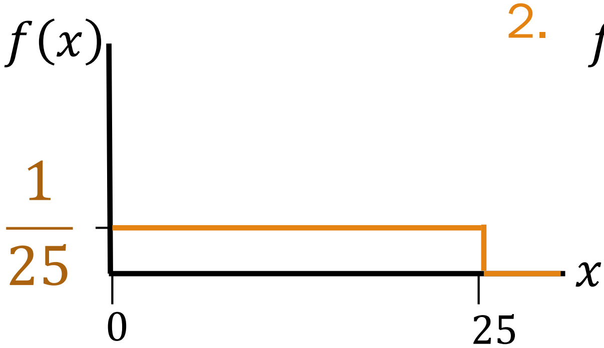
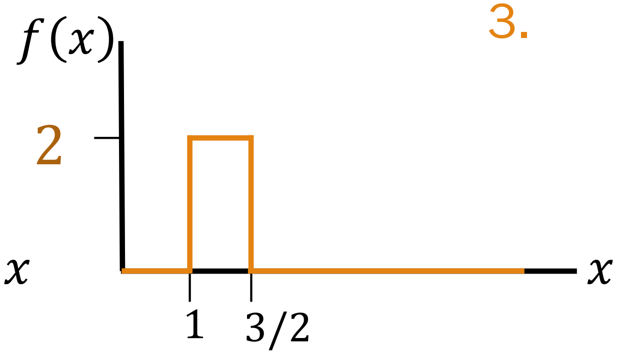


2.



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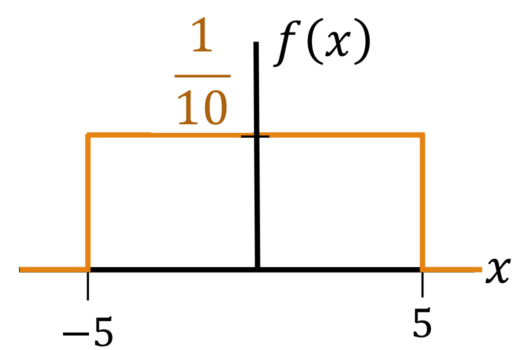


Expectation and Variance

<u>Discrete</u> RV X

$$E[X] = \sum_x x\, p(x)$$

$$E[g(X)] = \sum_x g(x)\, p(x)$$

<u>Continuous</u> RV X

$$E[X] = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx$$

<u>Both</u> continuous and discrete RVs

$$E[aX + b] = aE[X] + b$$

$$\text{Var}(X) = E[(X - E[X])^2] = E[X^2] - (E[X])^2$$

$$\text{Var}(aX + b) = a^2\text{Var}(X)$$

Linearity of Expectation

Properties of variance

TL;DR: $\sum_{x=a}^{b} \Rightarrow \int_a^b$

Uniform RV expectation

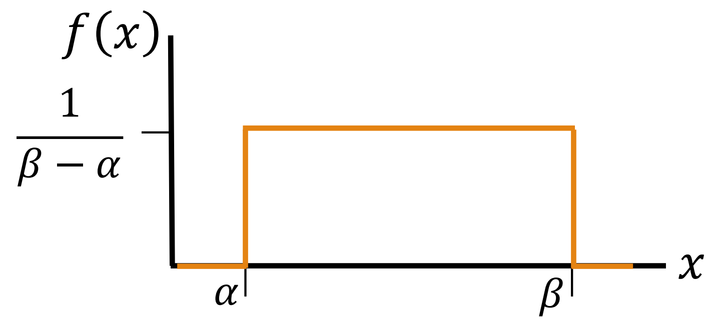
$$E[X] = \int_{-\infty}^{\infty} x \cdot f(x)dx$$

$$= \int_{\alpha}^{\beta} x \cdot \frac{1}{\beta - \alpha} dx$$

$$= \frac{1}{\beta - \alpha} \cdot \frac{1}{2} x^2 \Big|_{\alpha}^{\beta}$$

$$= \frac{1}{\beta - \alpha} \cdot \frac{1}{2} (\beta^2 - \alpha^2)$$

$$= \frac{1}{2} \cdot \frac{(\beta + \alpha)(\beta - \alpha)}{\beta - \alpha} = \frac{\alpha + \beta}{2}$$



Interpretation:
Average the start & end

Uniform Random Variable

def An **Uniform** random variable X is defined as follows:

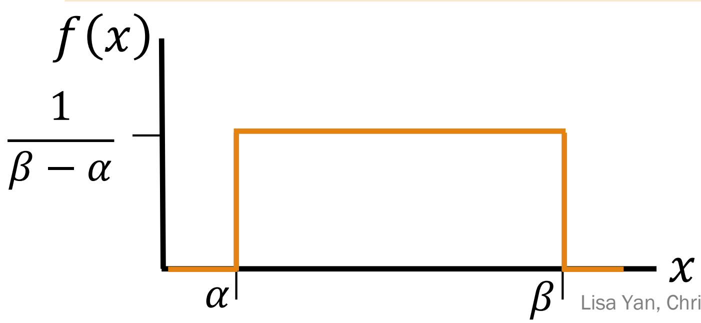
$$X \sim \text{Uni}(\alpha, \beta)$$

Support: $[\alpha, \beta]$
(sometimes defined over (α, β))

PDF $f(x) = \begin{cases} \dfrac{1}{\beta - \alpha} & \text{if } \alpha \le x \le \beta \\ 0 & \text{otherwise} \end{cases}$

Expectation $E[X] = \dfrac{\alpha + \beta}{2}$ Just now

Variance $\text{Var}(X) = \dfrac{(\beta - \alpha)^2}{12}$ On your own!



Exponential RV

Grid of random variables

	Number of successes	Time until success	
One trial	$\text{Ber}(p)$	$\text{Geo}(p)$	One success
	$n = 1$	$r = 1$	
Several trials	$\text{Bin}(n, p)$	$\text{NegBin}(r, p)$	Several successes
Interval of time	$\text{Poi}(\lambda)$	$\text{Exp}(\lambda)$	Amount of time before first success

Exponential Random Variable

Consider an experiment that lasts a duration of time until success occurs.

def An **Exponential** random variable X is the amount of time until success.

$X \sim \text{Exp}(\lambda)$

Support: $[0, \infty)$

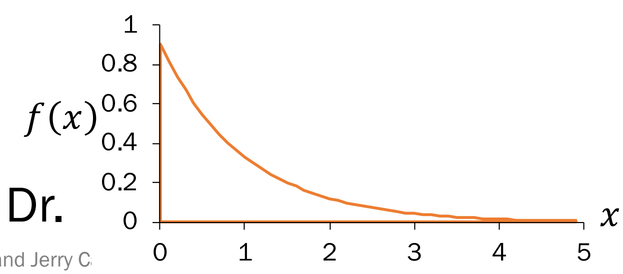
PDF $f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$

Expectation $E[X] = \frac{1}{\lambda}$ (in extra slides)

Variance $\text{Var}(X) = \frac{1}{\lambda^2}$ (on your own)

Examples:

- Time until next earthquake
- Time for request to reach web server
- Time until water main break on Campus Dr.



Earthquakes

ILL. No. 65. MEMORIAL ARCH, WITH CHURCH IN BACKGROUND, STANFORD UNIVERSITY, SHOWING TYPES OF CARVED WORK WITH THE SANDSTONE.

1906 Earthquake
Magnitude 7.8

Earthquakes

$X \sim \text{Exp}(\lambda)$ $E[X] = 1/\lambda$
$f(x) = \lambda e^{-\lambda x}$ if $x \geq 0$

Major earthquakes (magnitude 8.0+) occur once every 500 years.*

1. What is the probability of a major earthquake in the next 30 years?

Define events/ RVs & state goal

X: when next earthquake happens

$X \sim \text{Exp}(\lambda = 0.002)$

λ: year^{-1} $= 1/500$

Want: $P(X < 30)$

Solve

$$P(X<30) = \int_0^{30} 0.002\, e^{-0.002x}\, dx$$

$$= 0.002 \frac{-1}{0.002} e^{-0.002x} \Big|_0^{30}$$

$$= -\left(e^{-0.06} - e^{0.00}\right)$$

$$= 1 - e^{-0.06} \approx \boxed{0.058}$$

Recall

$$\int e^{cx} dx = \frac{1}{c} e^{cx}$$

*In California, according to historical data form USGS, 2015

Earthquakes

$X \sim \text{Exp}(\lambda)$ $E[X] = 1/\lambda$
$f(x) = \lambda e^{-\lambda x}$ if $x \geq 0$

Major earthquakes (magnitude 8.0+) occur once every 500 years.*

1. What is the probability of a major earthquake in the next 30 years?
2. What is the standard deviation of years until the next earthquake?

Define events/ RVs & state goal

X: when next earthquake happens

$X \sim \text{Exp}(\lambda = 0.002)$

λ: year^{-1}

Want: $P(X < 30)$

Solve

$$\text{Var}(X) = \frac{1}{\lambda^2} = \frac{1}{(0.002 \text{ year}^{-1})^2} = 250{,}000 \text{ years}^2$$

$$SD(X) = \sqrt{\text{Var}(X)} = \boxed{500 \text{ years}}$$

In general, $SD(X) = E[X] = \frac{1}{\lambda}$

whenever $X \sim \text{Exp}(\lambda)$

*In California, according to historical data form USGS, 2015

Cumulative Distribution Functions

Cumulative Distribution Function (CDF)

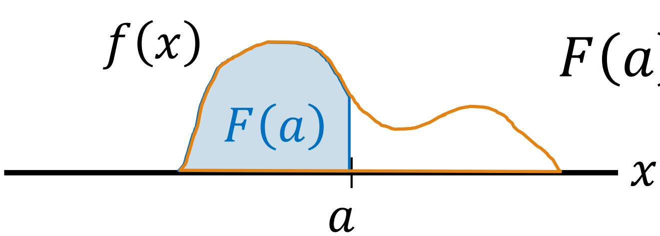
For a random variable X, the cumulative distribution function (CDF) is defined as

$$F(a) = F_X(a) = P(X \leq a), \text{where } -\infty < a < \infty$$

For a discrete RV X, the CDF is:

$$F(a) = P(X \leq a) = \sum_{\text{all } x \leq a} p(x)$$

For a continuous RV X, the CDF is:



$$F(a) = P(X \leq a) = \int_{-\infty}^{a} f(x)dx$$

CDF is a probability, though PDF is not.

If you learn to use CDFs, you can avoid integrating the PDF.

Using the CDF for continuous RVs

For a **continuous** random variable X with PDF $f(x)$, the CDF of X is

$$F(a) = P(X \leq a) = \int_{-\infty}^{a} f(x)dx$$

Matching (choices are used 0/1/2 times)

1. $P(X < a)$
2. $P(X > a)$
3. $P(X \geq a)$
4. $P(a \leq X \leq b)$

A. $F(a)$
B. $1 - F(a)$
C. $F(b) - F(a)$
D. $F(a) - F(b)$

Using the CDF for continuous RVs

For a **continuous** random variable X with PDF $f(x)$, the CDF of X is

$$F(a) = P(X \le a) = \int_{-\infty}^{a} f(x)dx$$

Matching (choices are used 0/1/2 times)

1. $P(X < a)$
2. $P(X > a)$
3. $P(X \ge a)$
4. $P(a \le X \le b)$

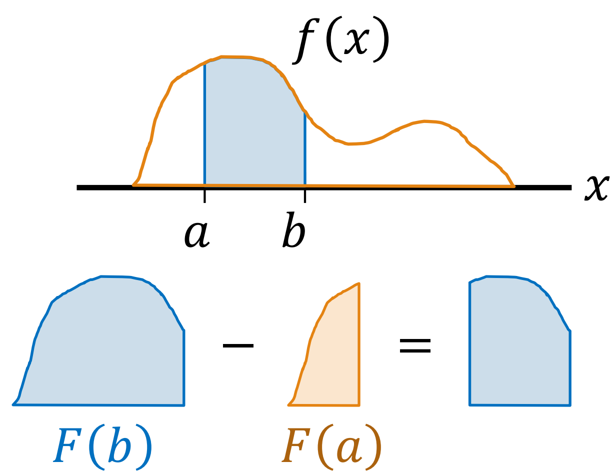
A. $F(a)$
B. $1 - F(a)$
C. $F(b) - F(a)$ (next slide)
D. $F(a) - F(b)$

Using the CDF for continuous RVs

For a continuous random variable X with PDF $f(x)$, the CDF of X is

$$F(a) = P(X \leq a) = \int_{-\infty}^{a} f(x)dx$$

4. $P(a \leq X \leq b) = F(b) - F(a)$



Proof:

$$F(b) - F(a) = \int_{-\infty}^{b} f(x)dx - \int_{-\infty}^{a} f(x)dx$$

$$= \left(\int_{-\infty}^{a} f(x)dx + \int_{a}^{b} f(x)dx\right) - \int_{-\infty}^{a} f(x)dx$$

$$= \int_{a}^{b} f(x)dx$$

CDF of an Exponential RV

$X \sim \text{Exp}(\lambda)$ $f(x) = \lambda e^{-\lambda x}$ if $x \geq 0$

$$X \sim \text{Exp}(\lambda) \qquad F(x) = 1 - e^{-\lambda x} \text{ if } x \geq 0$$

Proof:

remember that $f(y) = 0$ for $y < 0$

Recall

$$\int e^{cx} dx = \frac{1}{c} e^{cx}$$

$$F(x) = P(X \leq x) = \int_{y=-\infty}^{x} f(y) dy = \int_{y=0}^{x} \lambda e^{-\lambda y} dy$$

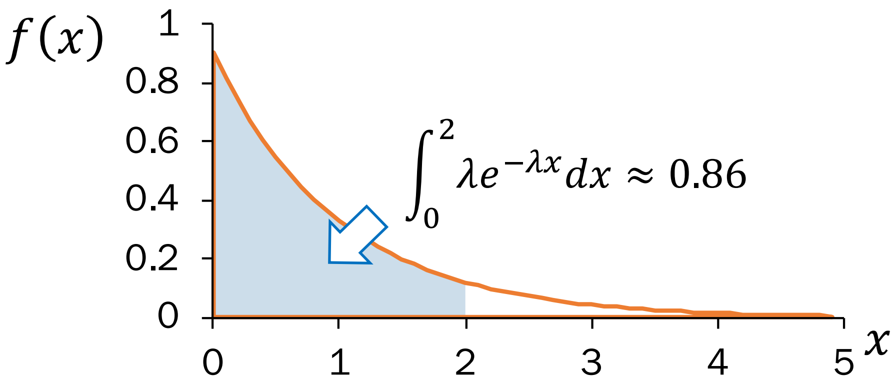
$$= \lambda \frac{1}{-\lambda} e^{-\lambda y} \Big|_0^x$$

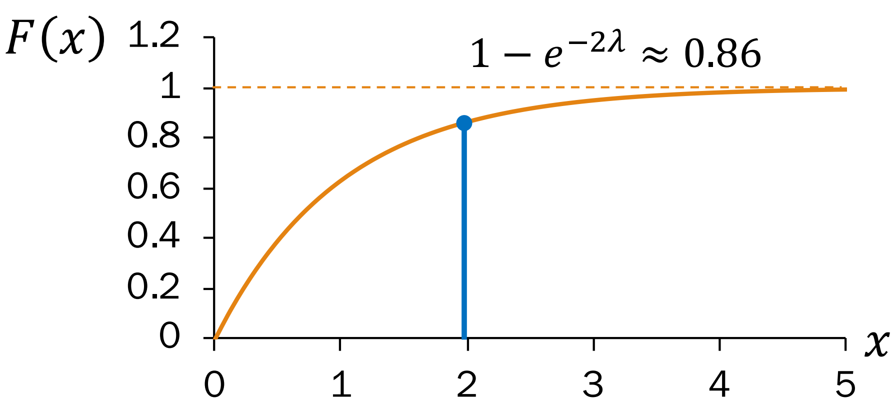
$$= -1\left(e^{-\lambda x} - e^{-\lambda 0}\right)$$

$$= 1 - e^{-\lambda x}$$

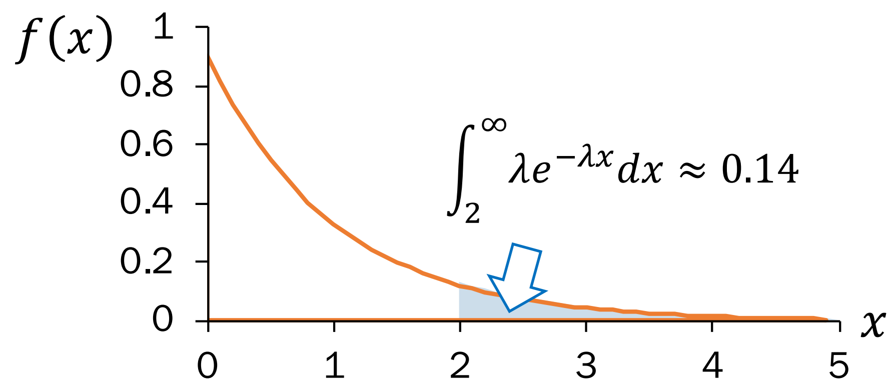
PDF/CDF $X \sim \text{Exp}(\lambda = 1)$

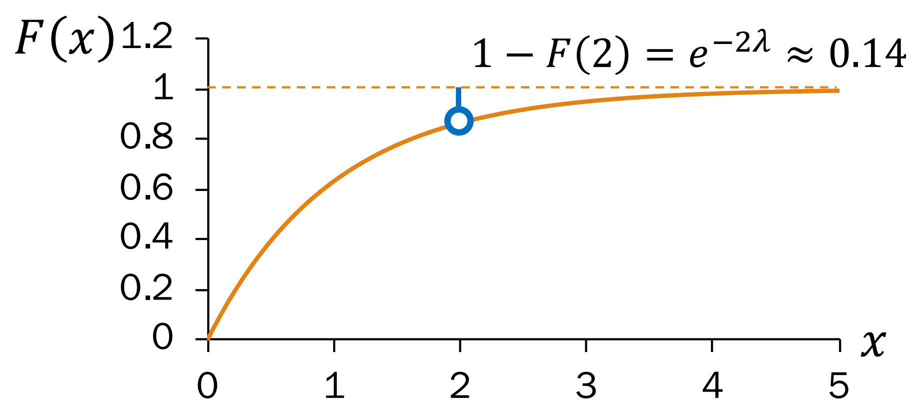
$X \sim \text{Exp}(\lambda)$ $\quad x \geq 0$: $\quad f(x) = \lambda e^{-\lambda x}$, $\quad F(x) = 1 - e^{-\lambda x}$





$$P(X \leq 2)$$





$$P(X > 2)$$

Memoryless Property

Memorylessness: Hurry Up and Wait

A continuous probability distribution is said to be **memoryless** if a random variable X on that probability distribution satisfies the following for all $s, t \geq 0$:

$$P(X \geq s + t \mid X \geq s) = P(X \geq t)$$

- Here, s represents the time you've already spent waiting.
- The above states that after you've waited s time units, the probability you'll need to wait an **additional** t time units is equal to the probability you'd have to wait t time units without having waited those s time units in the first place.
- Example: If train arrival is guided by a memoryless random variable, the fact that you've waited 15 minutes doesn't obligate the train to arrive any faster!

Memorylessness: Hurry Up and Wait

A continuous probability distribution is said to be **memoryless** if a random variable X on that probability distribution satisfies the following for all $s, t \geq 0$:

$$P(X \geq s + t \mid X \geq s) = P(X \geq t)$$

Using the definition of conditional probability, we can show that our Exponential distribution exhibits the memoryless property. Just let $X \sim \text{Exp}(\lambda)$ and trust the math:

$$P(X \geq s + t \mid X \geq s) = \frac{P(X \geq s + t)}{P(X \geq s)} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = P(X \geq t)$$

Exercises

Earthquakes

Major earthquakes (magnitude 8.0+) occur independently on average once every 500 years.*

What is the probability of zero major earthquakes next year?

*In California, according to historical data form USGS, 2015

Earthquakes

Major earthquakes (magnitude 8.0+) occur independently on average once every 500 years.*

What is the probability of zero major earthquakes next year?

Strategy 1: Exponential RV

Define events/RVs & state goal

T: when first earthquake happens

$T \sim \text{Exp}(\lambda = 0.002)$

Want: $P(T > 1) = 1 - F(1)$

Solve

$$P(T > 1) = 1 - \left(1 - e^{-\lambda \cdot 1}\right) = e^{-\lambda}$$

$$= e^{-0.002} \approx 0.998$$

*In California, according to historical data form USGS, 2015

Earthquakes

$Y \sim \text{Poi}(\lambda)$ $\quad p(k) = e^{-\lambda}\frac{\lambda^k}{k!}$

Major earthquakes (magnitude 8.0+) occur independently on average once every 500 years.*

What is the probability of zero major earthquakes next year?

Strategy 1: Exponential RV

Define events/RVs & state goal

T: when first earthquake happens

$T \sim \text{Exp}(\lambda = 0.002)$

Want: $P(T > 1) = 1 - F(1)$

Solve

$P(T > 1) = 1 - \left(1 - e^{-\lambda \cdot 1}\right) = e^{-\lambda}$

Strategy 2: Poisson RV

Define events/RVs & state goal

N: # earthquakes next year

$N \sim \text{Poi}(\lambda = 0.002)$

$\lambda: \frac{\text{earthquakes}}{\text{year}}$

Want: $P(N = 0)$

Solve

$$P(N = 0) = \frac{\lambda^0 e^{-\lambda}}{0!} = e^{-\lambda} \approx 0.998$$

*In California, according to historical data form USGS, 2015

Replacing your laptop

$X \sim \text{Exp}(\lambda)$ $\quad E[X] = 1/\lambda$ $\quad F(x) = 1 - e^{-\lambda x}$

Let $X =$ # hours of use until your laptop dies.

- X is distributed as an Exponential RV, where
- On average, laptops die after 5000 hours of use.
- You use your laptop 5 hours a day.

What is P(your laptop lasts 4 years)?

Replacing your laptop

$X \sim \text{Exp}(\lambda)$ $\quad E[X] = 1/\lambda$ $\quad F(x) = 1 - e^{-\lambda x}$

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Define

X: # hours until laptop death

$X \sim \text{Exp}(\lambda = 1/5000)$

Want: $P(X > 5 \cdot 365 \cdot 4)$

Solve

$$P(X > 7300) = 1 - F(7300)$$
$$= 1 - \left(1 - e^{-7300/5000}\right) = e^{-1.46} \approx 0.2322$$

Better plan ahead if you're co-terming!

- 5-year plan: $P(X > 9125) = e^{-1.825} \approx 0.1612$
- 6-year plan: $P(X > 10950) = e^{-2.19} \approx 0.1119$

Extra

Expectation of the Exponential

$X \sim \text{Exp}(\lambda)$ $f(x) = \lambda e^{-\lambda x}$ if $x \geq 0$

$X \sim \text{Exp}(\lambda)$ Expectation $E[X] = \frac{1}{\lambda}$

Proof:

$$E[X] = \int_{-\infty}^{\infty} x f(x) dx = \int_{0}^{\infty} x \lambda e^{-\lambda x} dx$$

$$= -x e^{-\lambda x} \Big|_0^{\infty} + \int_0^{\infty} e^{-\lambda x} dx$$

$$= -x e^{-\lambda x} \Big|_0^{\infty} - \frac{1}{\lambda} e^{-\lambda x} \Big|_0^{\infty}$$

$$= [0 - 0] + \left[0 - \left(\frac{-1}{\lambda}\right)\right]$$

$$= \frac{1}{\lambda}$$

Integration by parts

$$\int x \lambda e^{-\lambda x} dx = \int u \cdot dv$$

$u = x$ $\quad dv = \lambda e^{-\lambda x} dx$

$du = dx$ $\quad v = -e^{-\lambda x}$

$$\int u \cdot dv = u \cdot v - \int v \cdot du$$

$$-x e^{-\lambda x} - \int -e^{-\lambda x} dx$$