10: Normal Distributions

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Lecture Discussion on Ed

Normal Random Variables

Normal Random Variable

def A **Normal** random variable X is defined as follows:

$$X \sim \mathcal{N}(\mu, \sigma^2)$$

Support: $(-\infty, \infty)$

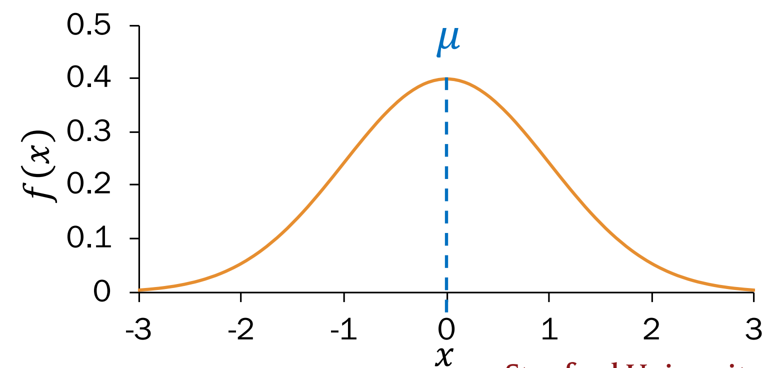
PDF	$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$
Expectation	$E[X] = \mu$
Variance	$\text{Var}(X) = \sigma^2$

Other names: **Gaussian** random variable

mean

variance

$$X \sim \mathcal{N}(\mu, \sigma^2)$$



Carl Friedrich Gauss

Carl Friedrich Gauss (1777-1855) was a remarkably influential German mathematician.

Johann Carl Friedrich Gauss (/gaʊs/; German: *Gauß* [gaʊs] listen); Latin: *Carolus Fridericus Gauss*; 30 April 1777 – 23 February 1855) was a German mathematician and physicist who made significant contributions to many fields, including algebra, analysis, astronomy, differential geometry, electrostatics, geodesy, geophysics, magnetic fields, matrix theory, mechanics, number theory, optics and statistics. } just wow!

Sometimes referred to as the *Princeps mathematicorum*[1] (Latin for "the foremost of mathematicians") and "the greatest mathematician since antiquity", Gauss had an exceptional influence in many fields of mathematics and science, and is ranked among history's most influential mathematicians.[2]

Did **not** invent Normal distribution but rather popularized it.

Why the Normal?

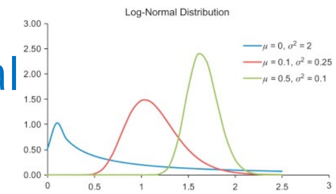
- Common for natural phenomena: height, weight, etc.
- Most noise in the world is Normal
- Often results from the sum of many random variables
- Sample means are distributed normally

That's what they want you to believe…

Why the Normal?

- Common for natural phenomena: height, weight, etc.

Actually log-normal



- Most noise in the world is Normal

Just an assumption

- Often results from the sum of many random variables

Only if equally weighted

- Sample means are distributed normally

(okay this one is true, we'll see this in 3 weeks)

Okay, so why the Normal?

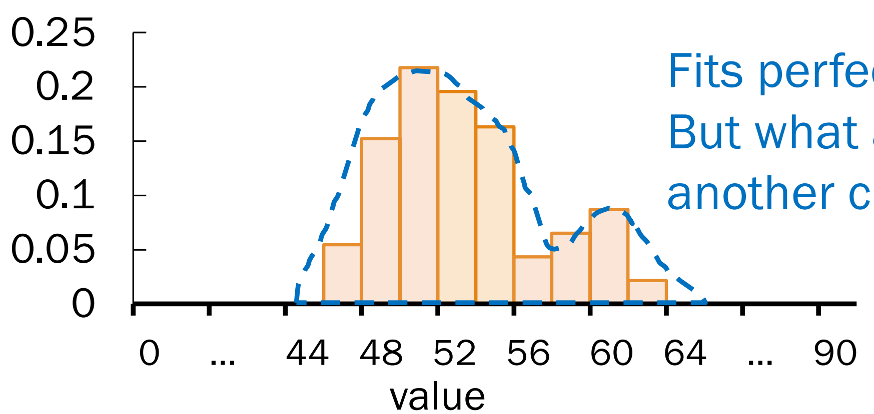
Part of CS109 learning goals:

- Translate a problem statement into a random variable

In other words: model real life situations with probability distributions

<u>How do you model student heights</u>?

- Suppose you have data from one classroom.



Fits perfectly!
But what about in
another classroom?

Okay, so why the Normal?

Part of CS109 learning goals:

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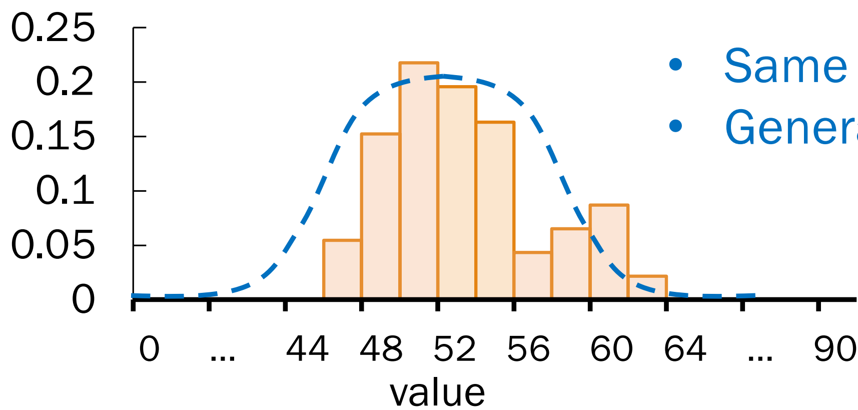
How do you model student heights?

- Suppose you have data from one classroom.



- Same mean/var
- Generalizes well

A Gaussian maximizes entropy for a given mean and variance.

Why the Normal?

- Common for natural phenomena: height, weight, etc.
- Most noise in the world is Normal
- Often results from ... random va...
- Sampl... ...ans are distributed normally

Actually log-norm...

...mption

Only if equally weighted

(okay this one is true, we'll see this in 3 weeks)

because it's well understood

Stay critical of how to model real-world phenomena.

Anatomy of a beautiful equation

Let $X \sim \mathcal{N}(\mu, \sigma^2)$.

The PDF of X is defined as:

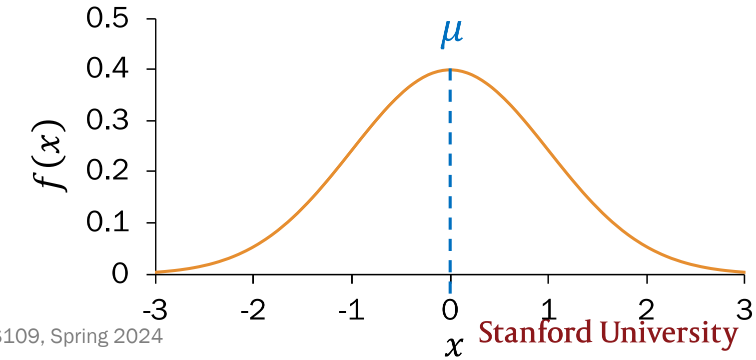
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

symmetric around μ

variance σ^2 manages spread

exponential tail

normalizing constant



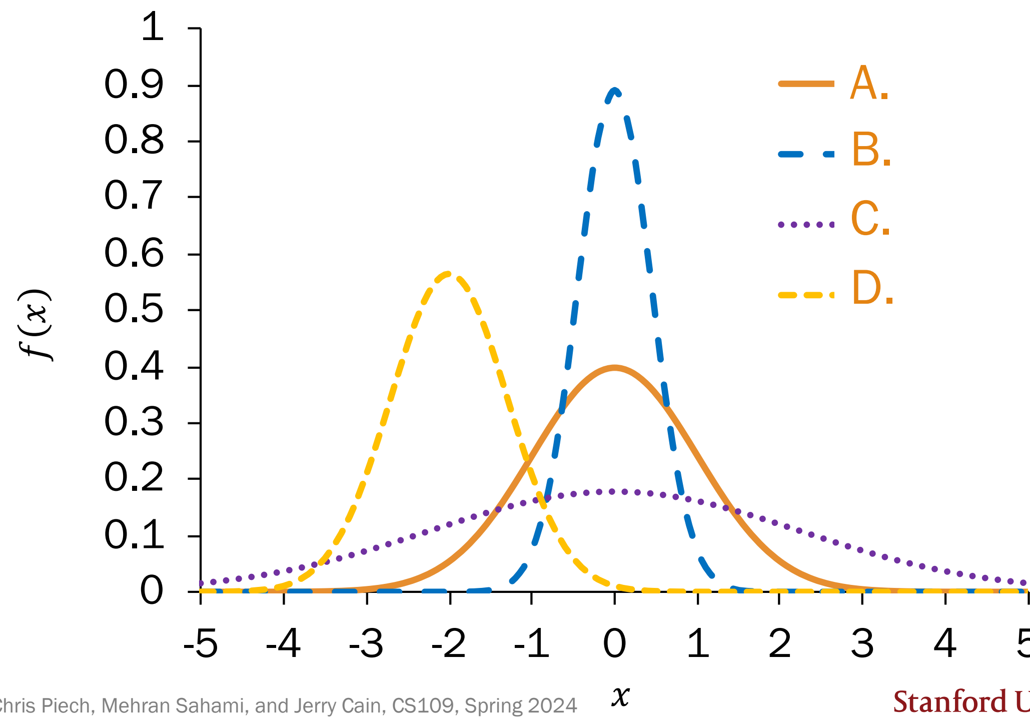
Normal Random Variable

$$X \sim \mathcal{N}(\mu, \sigma^2)$$

mean: μ, variance: σ^2

Match PDF to distribution:

1. $\mathcal{N}(0, 1)$
2. $\mathcal{N}(-2, 0.5)$
3. $\mathcal{N}(0, 5)$
4. $\mathcal{N}(0, 0.2)$



Getting to class

You spend some minutes, X, traveling between classes.

- Average time spent: $\mu = 4$ minutes
- Variance of time spent: $\sigma^2 = 2$ minutes^2

Suppose X is normally distributed. What is the probability you spend ≥ 6 minutes traveling?

$$X \sim \mathcal{N}(\mu = 4, \sigma^2 = 2)$$

$$P(X \geq 6) = \int_6^\infty f(x)dx = \int_6^\infty \frac{1}{2\sqrt{\pi}} e^{-\frac{(x-4)^2}{4}} dx$$

(tell Jerry if you solve this analytically and we'll be famous together)

Love and Anger in the Same Formula

Computing probabilities with Normal RVs

For a Normal RV $X \sim \mathcal{N}(\mu, \sigma^2)$, its CDF has no closed form.

$$P(X \leq x) = F(x) = \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy$$

⚠ Cannot be solved analytically

However, we can solve for probabilities numerically using a function Φ:

$$F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

CDF of $X \sim \mathcal{N}(\mu, \sigma^2)$

A function that has been solved **numerically**

To get here, we'll first need to know some properties of Normal RVs.

Normal RV: Properties

Properties of Normal RVs

Let $X \sim \mathcal{N}(\mu, \sigma^2)$ with CDF $P(X \leq x) = F(x)$.

1. Linear transformations of Normal RVs are also Normal RVs.

   If $Y = aX + b$, then $Y \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$.

2. The PDF of a Normal RV is symmetric about the mean μ.

$$F(\mu - x) = 1 - F(\mu + x)$$

1. Linear transformations of Normal RVs

Let $X \sim \mathcal{N}(\mu, \sigma^2)$ with CDF $P(X \leq x) = F(x)$.

Linear transformations of X are also Normal.

If $Y = aX + b$, then $Y \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$

Proof:

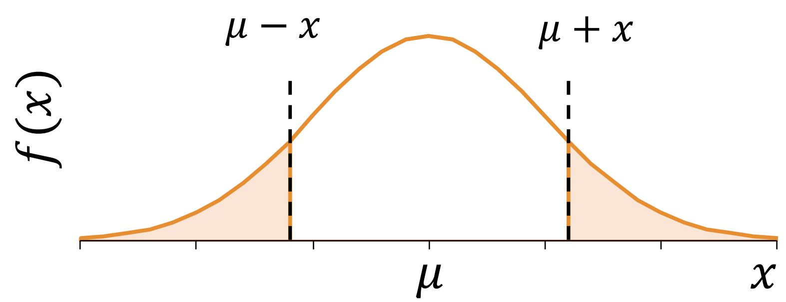
- $E[Y] = E[aX + b] = aE[X] + b = a\mu + b$ — Linearity of Expectation
- $\text{Var}(Y) = \text{Var}(aX + b) = a^2\text{Var}(X) = a^2\sigma^2$ — $\text{Var}(aX + b) = a^2\text{Var}(X)$
- Y is also Normal — Proof in Ross, 10th ed (Section 5.4)

2. Symmetry of Normal RVs

Let $X \sim \mathcal{N}(\mu, \sigma^2)$ with CDF $P(X \leq x) = F(x)$.

The PDF of a Normal RV is symmetric about the mean μ.

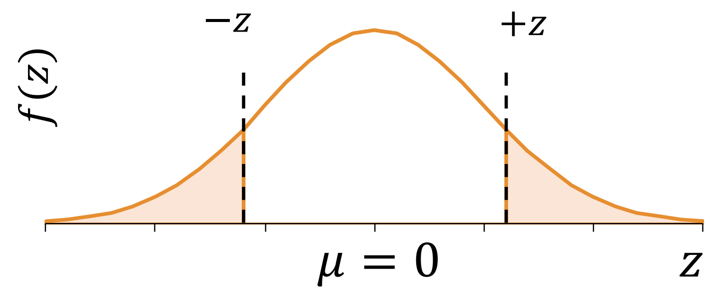
$$F(\mu - x) = 1 - F(\mu + x)$$



Using symmetry of the Normal RV

$F(\mu - x) = 1 - F(\mu + x)$

Let $Z \sim \mathcal{N}(0,1)$ with CDF $P(Z \le z) = F(z)$.



Suppose we only knew numeric values for $F(z)$ and $F(y)$, for some $y, z \ge 0$.

How do we compute the following probabilities?

1. $P(Z \le z)$ $= F(z)$
2. $P(Z < z)$
3. $P(Z \ge z)$
4. $P(Z \le -z)$
5. $P(Z \ge -z)$
6. $P(y < Z < z)$

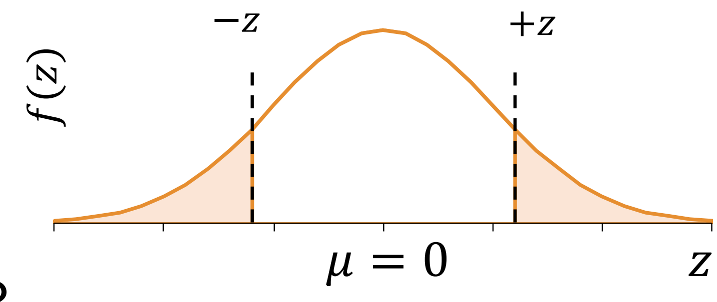
A. $F(z)$
B. $1 - F(z)$
C. $F(z) - F(y)$

Using symmetry of the Normal RV

$F(\mu - x) = 1 - F(\mu + x)$

Let $Z \sim \mathcal{N}(0,1)$ with CDF $P(Z \leq z) = F(z)$.

Suppose we only knew numeric values for $F(z)$ and $F(y)$, for some $y, z \geq 0$.



How do we compute the following probabilities?

1. $P(Z \leq z) \quad = F(z)$
2. $P(Z < z) \quad = F(z)$
3. $P(Z \geq z) \quad = 1 - F(z)$
4. $P(Z \leq -z) \quad = 1 - F(z)$
5. $P(Z \geq -z) \quad = F(z)$
6. $P(y < Z < z) \quad = F(z) - F(y)$

A. $F(z)$
B. $1 - F(z)$
C. $F(z) - F(y)$

Symmetry is particularly useful when computing probabilities of zero-mean Normal RVs.

Normal RV: Computing probability

Computing probabilities with Normal RVs

Let $X \sim \mathcal{N}(\mu, \sigma^2)$.

To compute the CDF, $P(X \leq x) = F(x)$:

- We cannot analytically solve the integral, as it has no closed form.
- … but we ***can*** solve numerically using a function Φ:

$$F(x) = \Phi\left(\frac{x - \mu}{\sigma}\right)$$

CDF of the Standard Normal, Z

Standard Normal RV, Z

The Standard Normal random variable Z is defined as follows:

$Z \sim \mathcal{N}(0, 1)$	Expectation	$E[Z] = \mu = 0$
	Variance	$\text{Var}(Z) = \sigma^2 = 1$

Note: not a new distribution; just a special case of the Normal

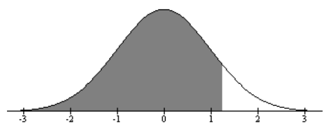
Other names: Unit Normal

CDF of Z defined as: $P(Z \leq z) = \Phi(z)$

Φ has been numerically computed

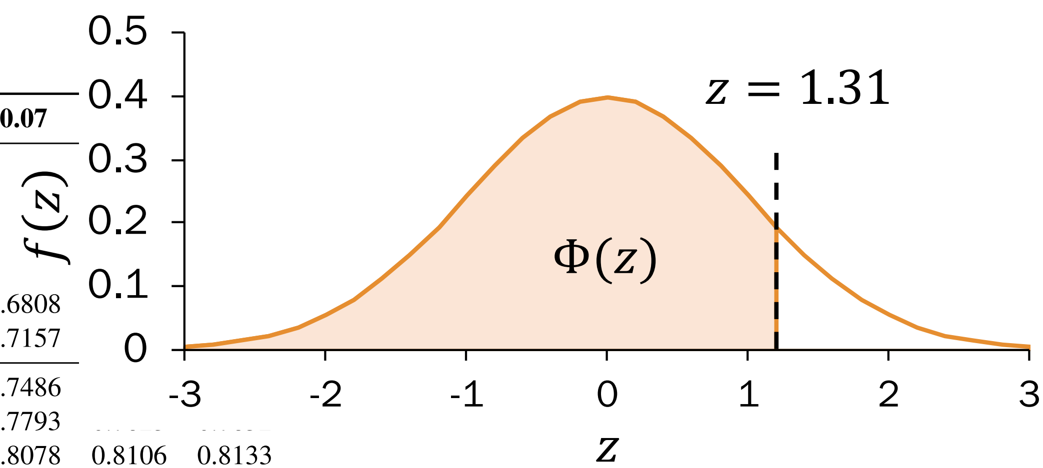
Standard Normal Table

An entry in the table is the area under the curve to the left of z, $P(Z \le z) = \Phi(z)$.



Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07		
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0		
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0		
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0		
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0		
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808		
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157		
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486		
0.7	0.7580	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7793		
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8906	0.8925	0.8943	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441

$$P(Z \le 1.31) = \Phi(1.31)$$



Standard Normal Table only has probabilities $\Phi(z)$ for $z \ge 0$.

History fact: Standard Normal Table

TABLES

SERVANT

AU CALCUL DES RÉFRACTIONS

APPROCHANTES DE L'HORIZON.

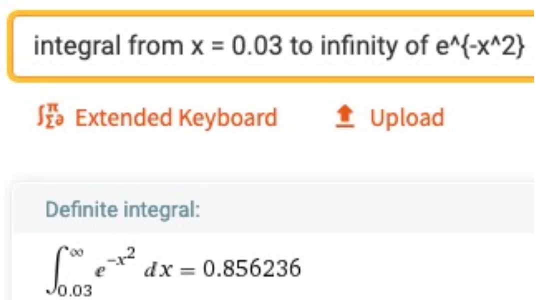
TABLE PREMIÈRE.

Intégrales de $e^{-tt}\,dt$, depuis une valeur quelconque de t jusqu'à t infinie.

t	*Intégrale.*	*Diff. prem.*	*Diff. II.*	*Diff. III.*
0,00	0,88622692	999968	201	199
0,01	0,87622724	999767	400	199
0,02	0,86622957	999367	599	200
0,03	0,85623590	998768	799	199
0,04	0,84624822	997969	998	197
0,05	0,83626853	996971	1195	199
0,06	0,82629882	995776	1394	196

The first Standard Normal Table was computed by Christian Kramp, French astronomer (1760–1826), in *Analyse des Réfractions Astronomiques et Terrestres*, 1799

Used a Taylor series expansion to the third power



Probabilities for a general Normal RV

Let $X \sim \mathcal{N}(\mu, \sigma^2)$. To compute the CDF $P(X \le x) = F(x)$, we use Φ, the CDF for the Standard Normal $Z \sim \mathcal{N}(0, 1)$:

$$F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

Proof:

$$F(x) = P(X \le x)$$ Definition of CDF

$$= P(X - \mu \le x - \mu) = P\left(\frac{X-\mu}{\sigma} \le \frac{x-\mu}{\sigma}\right)$$ Algebra + $\sigma > 0$

$$= P\left(Z \le \frac{x-\mu}{\sigma}\right)$$

$$= \Phi\left(\frac{x-\mu}{\sigma}\right)$$

- $\frac{X-\mu}{\sigma} = \frac{1}{\sigma}X - \frac{\mu}{\sigma}$ is a linear transform of X.
- This is distributed as $\mathcal{N}\left(\frac{1}{\sigma}\mu - \frac{\mu}{\sigma}, \frac{1}{\sigma^2}\sigma^2\right) = \mathcal{N}(0,1)$
- In other words, $\frac{X-\mu}{\sigma} = Z \sim \mathcal{N}(0,1)$ with CDF Φ.

Probabilities for a general Normal RV

Let $X \sim \mathcal{N}(\mu, \sigma^2)$. To compute the CDF $P(X \leq x) = F(x)$, we use Φ, the CDF for the Standard Normal $Z \sim \mathcal{N}(0, 1)$:

$$F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

Proof:

$$F(x) = P(X \leq x) \qquad \text{Definition of CDF}$$

$$= P(X - \mu \leq x - \mu) = P\left(\frac{X-\mu}{\sigma} \leq \frac{x-\mu}{\sigma}\right) \qquad \text{Algebra} + \sigma > 0$$

$$= P\left(Z \leq \frac{x-\mu}{\sigma}\right)$$

$$= \Phi\left(\frac{x-\mu}{\sigma}\right)$$

- $\frac{X-\mu}{\sigma} = \frac{1}{\sigma}X - \frac{\mu}{\sigma}$ is a linear transform of X.

1. Compute $z = (x - \mu)/\sigma$.
2. Look up $\Phi(z)$ in Standard Normal table.

Campus bikes

You spend some minutes, X, traveling between classes.

- Average time spent: $\mu = 4$ minutes
- Variance of time spent: $\sigma^2 = 2$ minutes^2

Suppose X is normally distributed. What is the probability you spend ≥ 6 minutes traveling?

$X \sim \mathcal{N}(\mu = 4, \sigma^2 = 2)$

✕ $P(X \geq 6) = \int_6^\infty f(x)dx$ (no analytic solution)

1. Compute $z = \frac{(x-\mu)}{\sigma}$

$$P(X \geq 6) = 1 - F_x(6)$$
$$= 1 - \Phi\left(\frac{6-4}{\sqrt{2}}\right)$$
$$\approx 1 - \Phi(1.41)$$

2. Look up $\Phi(z)$ in table

$$1 - \Phi(1.41)$$
$$\approx 1 - 0.9207$$
$$= 0.0793$$

Is there an easier way? (yes)

Let $X \sim \mathcal{N}(\mu, \sigma^2)$. What is $P(X \leq x) = F(x)$?

- Use Python

```
from scipy import stats
X = stats.norm(mu, std)
X.cdf(x)
```

SciPy reference:
https://docs.scipy.org/doc/scipy/reference/generated/scipy.stats.norm.html

I'm not sure why Python decided to parameterize `stats.norm` around the standard deviation instead of the variance, but it did. ☺

Exercises

Get your Gaussian On

Let $X \sim \mathcal{N}(\mu = 3, \sigma^2 = 16)$. Std deviation $\sigma = 4$.

1. $P(X > 0)$

- If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$
- Symmetry of the PDF of Normal RV implies $\Phi(-z) = 1 - \Phi(z)$

Get your Gaussian On

Let $X \sim \mathcal{N}(\mu = 3, \sigma^2 = 16)$.
Note standard deviation $\sigma = 4$.

How would you write each of the below probabilities as a function of the standard normal CDF, Φ?

- If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$
- Symmetry of the PDF of Normal RV implies $\Phi(-z) = 1 - \Phi(z)$

1. $P(X > 0)$
2. $P(2 < X < 5)$
3. $P(|X - 3| > 6)$

Get your Gaussian On

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- If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$
- Symmetry of the PDF of Normal RV implies $\Phi(-x) = 1 - \Phi(x)$

Compute $z = \frac{(x-\mu)}{\sigma}$

Look up $\Phi(z)$ in table

$$P(X < -3) + P(X > 9)$$

$$= F(-3) + \left(1 - F(9)\right)$$

$$= \Phi\left(\frac{-3-3}{4}\right) + \left(1 - \Phi\left(\frac{9-3}{4}\right)\right)$$

Get your Gaussian On

Let $X \sim \mathcal{N}(\mu = 3, \sigma^2 = 16)$. Std deviation $\sigma = 4$.

1. $P(X > 0)$
2. $P(2 < X < 5)$
3. $P(|X - 3| > 6)$

- If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$
- Symmetry of the PDF of Normal RV implies $\Phi(-x) = 1 - \Phi(x)$

Compute $z = \frac{(x-\mu)}{\sigma}$

$$P(X < -3) + P(X > 9)$$

$$= F(-3) + (1 - F(9))$$

$$= \Phi\left(\frac{-3-3}{4}\right) + \left(1 - \Phi\left(\frac{9-3}{4}\right)\right)$$

Look up $\Phi(z)$ in table

$$= \Phi\left(-\frac{3}{2}\right) + \left(1 - \Phi\left(\frac{3}{2}\right)\right)$$

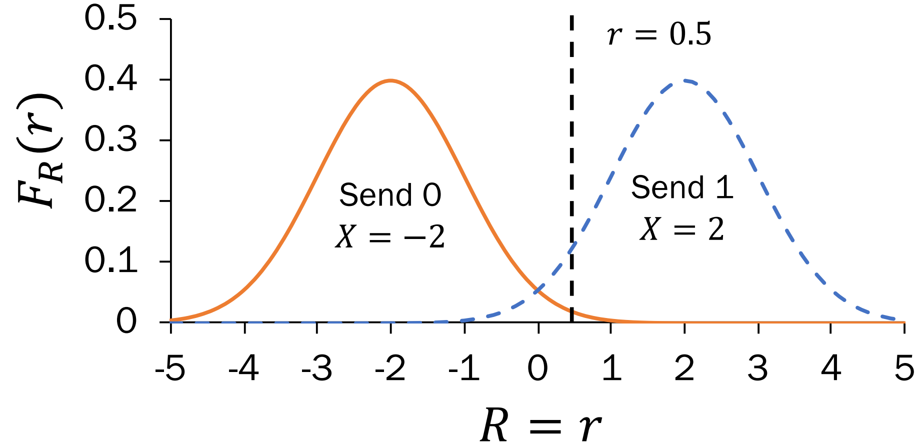
$$= 2\left(1 - \Phi\left(\frac{3}{2}\right)\right)$$

$$\approx 0.1337$$

Noisy Wires

Send a voltage of 2 V or -2 V on wire (to denote 1 and 0, respectively).

- X = voltage sent (2 or -2)
- Y = noise, $Y \sim \mathcal{N}(0, 1)$
- $R = X + Y$ voltage received.



Decode: $\begin{cases} 1 & \text{if } R \geq 0.5 \\ 0 & \text{otherwise.} \end{cases}$

1. What is P(decoding error | original bit is 1)?
   i.e., we sent 1, but we decoded as 0?
2. What is P(decoding error | original bit is 0)?

These probabilities are unequal. Why might this be useful?

Noisy Wires

Send a voltage of 2 V or -2 V on wire (to denote 1 and 0, respectively).

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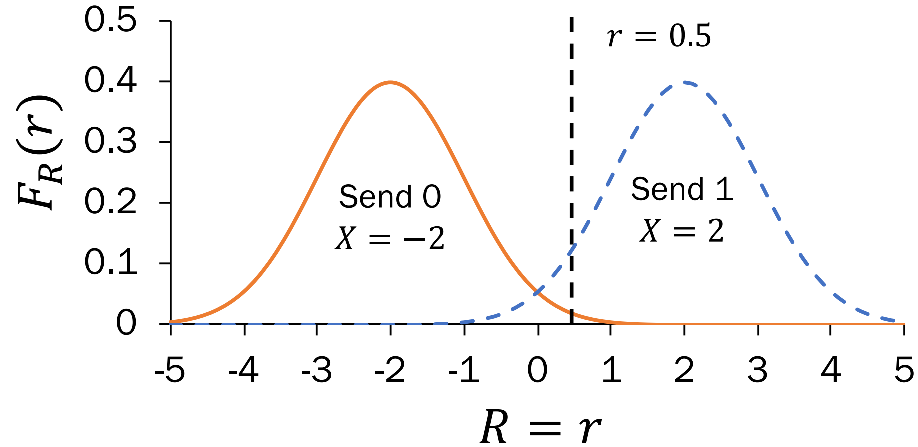

Decode: 1 if $R \geq 0.5$
0 otherwise.

1. What is P(decoding error | original bit is 1)?
   i.e., we sent 1, but we decoded as 0?

$$P(R < 0.5 | X = 2) = P(2 + Y < 0.5) = P(Y < -1.5)$$ Y is Standard Normal

$$= \Phi(-1.5) = 1 - \Phi(1.5) \approx 0.0668$$

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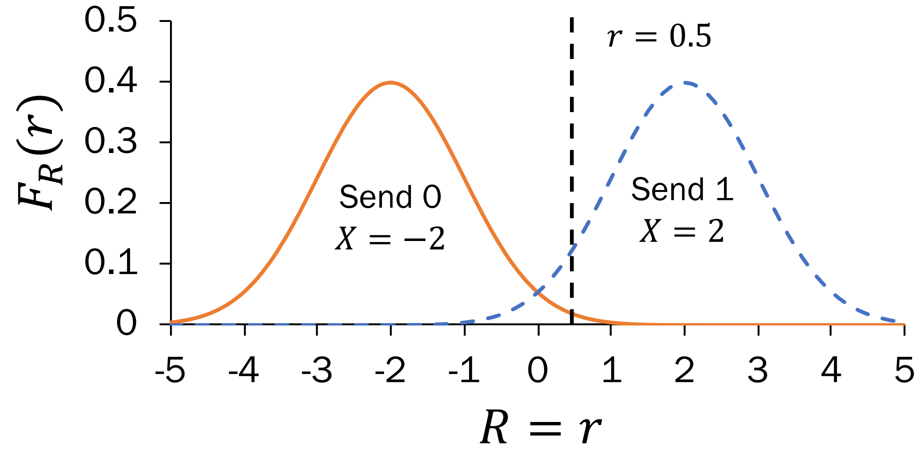


Decode: 1 if $R \geq 0.5$
0 otherwise.

1. What is P(decoding error | original bit is 1)? i.e., we sent 1, but we decoded as 0?

0.0668

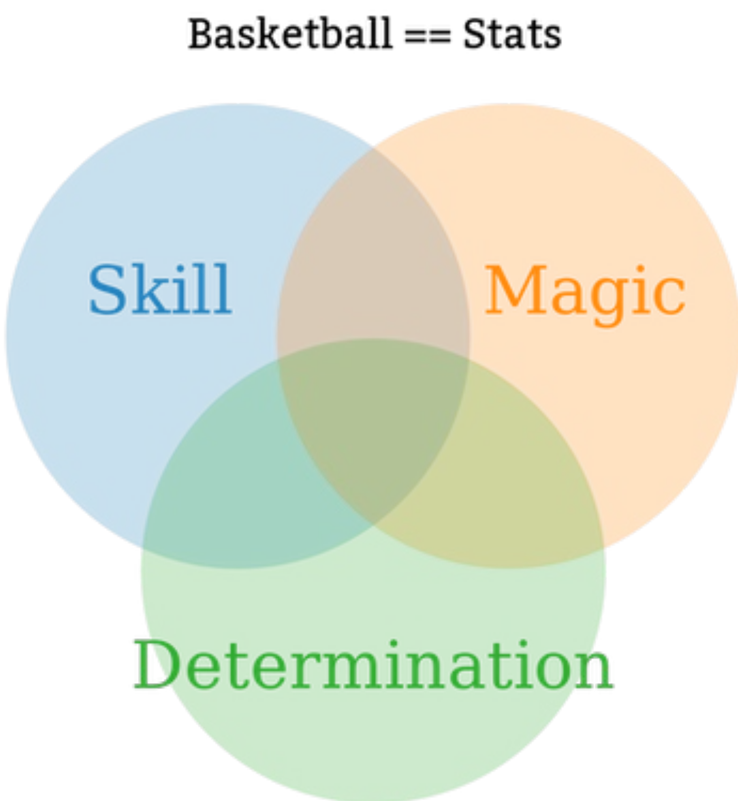
2. What is P(decoding error | original bit is 0)?

$$P(R \geq 0.5 | X = -2) = P(-2 + Y \geq 0.5) = P(Y \geq 2.5) \approx 0.0062$$

Asymmetric decoding probability: We would like to avoid mistaking a 0 for 1. Errors the other way are tolerable.

Sampling with the Normal RV

ELO ratings



What is the probability that the Warriors win?
More generally: How can you model zero-sum games?

ELO ratings

Each team has an ELO score S, calculated based on its past performance.

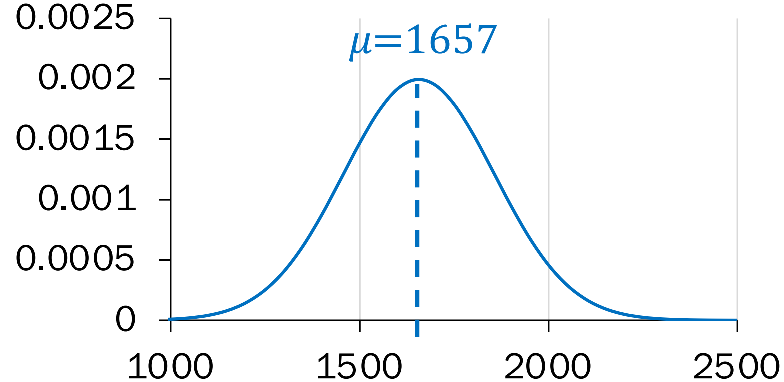
- Each game, a team has ability $A \sim \mathcal{N}(S, 200^2)$.
- The team with the higher sampled ability wins.


Arpad Elo

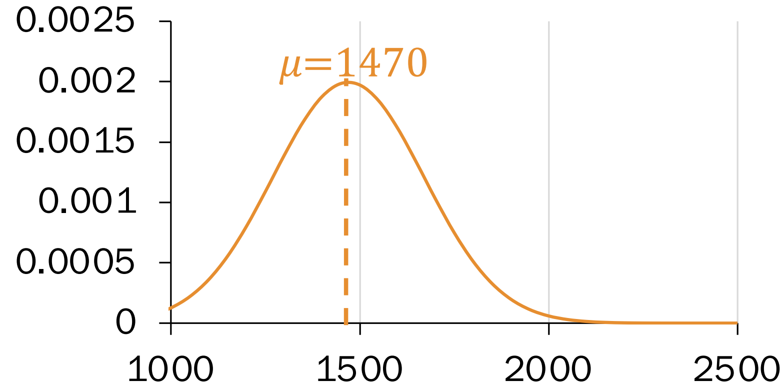
What is the probability that Warriors win this game?

Want: $P(\text{Warriors win}) = P(A_W > A_O)$

Warriors $A_W \sim \mathcal{N}(S = 1657, 200^2)$



Opponents $A_O \sim \mathcal{N}(S = 1470, 200^2)$



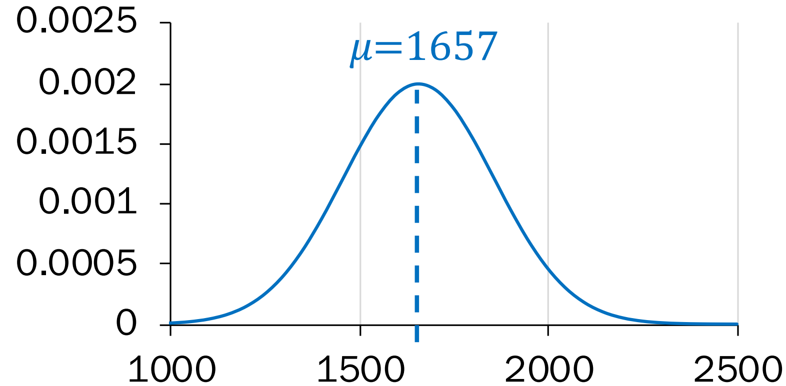
ELO ratings

Want: $P(\text{Warriors win}) = P(A_W > A_O)$

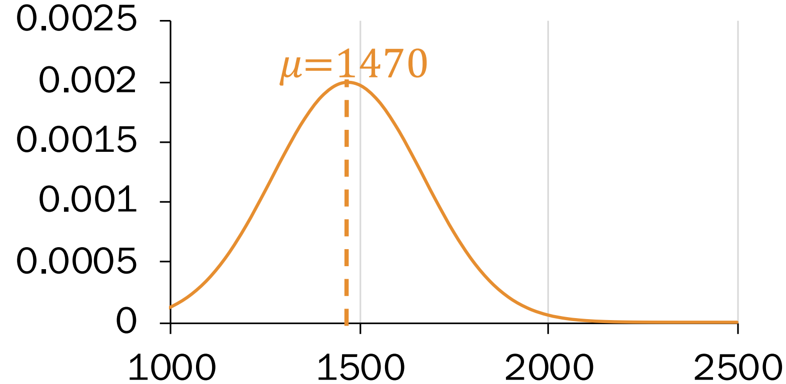
```python
from scipy import stats
WARRIORS_ELO = 1657
OPPONENT_ELO = 1470
STDEV = 200
NTRIALS = 10000

nSuccess = 0
for i in range(NTRIALS):
  w = stats.norm.rvs(WARRIORS_ELO, STDEV)
  o = stats.norm.rvs(OPPONENT_ELO, STDEV)
  if w > o: nSuccess += 1

print("Warriors sampled win fraction",
      float(nSuccess) / NTRIALS)
```

≈ 0.7488, calculated by sampling

Warriors $A_W \sim \mathcal{N}(S = 1657, 200^2)$



Opponents $A_O \sim \mathcal{N}(S = 1470, 200^2)$



Is there a better way?

$$P(A_W > A_O)$$

actual depiction of someone understanding joint continuous random variables

- This is a probability of an event involving **two continuous random variables!**
- We'll solve this problem analytically in less than two weeks' time.

Big goal for next lecture: Events involving two **discrete** random variables.

Stay tuned!