14: Conditional Expectation

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Lecture Discussion on Ed

Discrete conditional distributions

Discrete conditional distributions

Recall the definition of the conditional probability of event E given event F:

$$P(E|F) = \frac{P(EF)}{P(F)}$$

For discrete random variables X and Y, the conditional PMF of X given Y is

$$P(X = x|Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)}$$

Different notation, same idea:

$$p_{X|Y}(x|y) = \frac{p_{X,Y}(x, y)}{p_Y(y)}$$

Discrete probabilities of CS109

Each student responds with:

Year Y

- 1: Freshmen and Sophomores
- 2: Juniors and Seniors
- 3: Graduate Students and SCPD

Mood T:

- -1: 😕
- 0: 😐
- 1: 🥰

<u>Joint PMF</u>

	$Y=1$	$Y=2$	$Y=3$
$T=-1$	.06	.01	.01
$T=0$	.29	.14	.09
$T=1$	.30	.08	.02

$P(Y=2, T=1)$

Joint PMFs sum to 1.

Discrete probabilities of CS109

The below are **conditional probability tables** for conditional PMFs
(A) $P(Y = y|T = t)$ and (B) $P(T = t|Y = y)$.

1. Which is which?
2. What's the missing probability?

Joint PMF

	$Y = 1$	$Y = 2$	$Y = 3$
$T = -1$	.06	.01	.01
$T = 0$	.29	.14	.09
$T = 1$	.30	.08	.02

	$Y = 1$	$Y = 2$	$Y = 3$
$T = -1$	.09	.04	.08
$T = 0$	.45	.61	.75
$T = 1$	.46	.35	.17

	$Y = 1$	$Y = 2$	$Y = 3$
$T = -1$	.75	.125	?
$T = 0$	.56	.27	.17
$T = 1$	.75	.2	.05

Discrete probabilities of CS109

The below are **conditional probability tables** for conditional PMFs (A) $P(Y = y|T = t)$ and (B) $P(T = t|Y = y)$.

1. Which is which?
2. What's the missing probability?

Joint PMF

	$Y = 1$	$Y = 2$	$Y = 3$
$T = -1$	.06	.01	.01
$T = 0$	.29	.14	.09
$T = 1$	.30	.08	.02

(B) $P(T = t|Y = y)$

	$Y = 1$	$Y = 2$	$Y = 3$
$T = -1$	.09	.04	.08
$T = 0$	.45	.61	.75
$T = 1$	.46	.35	.17

.30/(.06+.29+.30)

(A) $P(Y = y|T = t)$

	$Y = 1$	$Y = 2$	$Y = 3$	
$T = -1$	.75	.125	.125	1-.75-.125
$T = 0$	.56	.27	.17	
$T = 1$	.75	.2	.05	

Conditional PMFs also sum to 1 conditioned on different events!

Quick check

$$P(X = x|Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)}$$

Number or function?

1. $P(X = 2|Y = 5)$
2. $P(X = x|Y = 5)$
3. $P(X = 2|Y = y)$
4. $P(X = x|Y = y)$

True or false?

5. $\sum_x P(X = x|Y = 5) = 1$
6. $\sum_y P(X = 2|Y = y) = 1$
7. $\sum_x \sum_y P(X = x|Y = y) = 1$
8. $\sum_x \left(\sum_y P(X = x|Y = y) P(Y = y) \right) = 1$

Quick check

$$P(X = x|Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)}$$

Number or function?

1. $P(X = 2|Y = 5)$

   number

2. $P(X = x|Y = 5)$

   1-D function

3. $P(X = 2|Y = y)$

   1-D function

4. $P(X = x|Y = y)$

   2-D function

True or false?

5. $\sum_x P(X = x|Y = 5) = 1$ true

6. $\sum_y P(X = 2|Y = y) = 1$ false

7. $\sum_x \sum_y P(X = x|Y = y) = 1$ false

8. $\sum_x \left(\sum_y P(X = x|Y = y) P(Y = y) \right) = 1$ true

Conditional Expectation

Conditional expectation

Recall the the conditional PMF of X given $Y = y$:

$$p_{X|Y}(x|y) = P(X = x|Y = y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$$

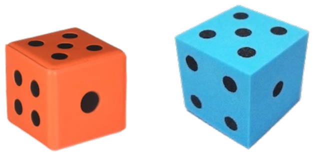
The conditional expectation of X given $Y = y$ is

$$E[X|Y = y] = \sum_x xP(X = x|Y = y) = \sum_x xp_{X|Y}(x|y)$$

- Note that $E[X]$ is a well-defined statistic even when X is one of many random variables in a multivariate distribution: $E[X] = \sum_x \sum_y xp_{X,Y}(x,y)$
- $E[X|Y = y]$ is the average value of X when Y is constrained to take on a specific value of y: $E[X|Y = y] = \sum_x xp_{X,Y}(x|y)$

It's been so long, our dice friends

$$E[X|Y=y] = \sum_x x p_{X|Y}(x|y)$$



- Roll two 6-sided dice.
- Let roll 1 be D_1, roll 2 be D_2.
- Let $S =$ value of $D_1 + D_2$.

1. What is $E[S|D_2 = 6]$?

$$E[S|D_2 = 6] = \sum_x x P(S = x|D_2 = 6)$$

$$= \left(\frac{1}{6}\right)(7 + 8 + 9 + 10 + 11 + 12)$$

$$= \frac{57}{6} = 9.5$$

Intuitively: $6 + E[D_1] = 6 + 3.5 = 9.5$ We'll prove in a moment

Properties of conditional expectation

1. LOTUS:

$$E[g(X)|Y=y] = \sum_x g(x) p_{X|Y}(x|y)$$

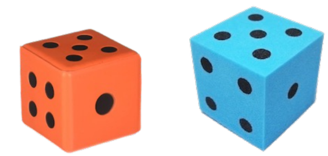
2. Linearity of conditional expectation:

$$E\left[\sum_{i=1}^{n} X_i \mid Y=y\right] = \sum_{i=1}^{n} E[X_i|Y=y]$$

3. Law of total expectation (in, like, three slides)

It's been so long, our dice friends

$$E[X|Y=y] = \sum_x x p_{X|Y}(x|y)$$

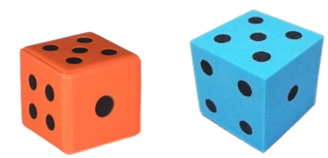


- Roll two 6-sided dice.
- Let roll 1 be D_1, roll 2 be D_2.
- Let $S =$ value of $D_1 + D_2$.

1. What is $E[S|D_2 = 6]$? $\quad \frac{57}{6} = 9.5$
2. What is $E[S|D_2]$?
   A. A function of S
   B. A function of D_2
   C. A number
3. Give an expression for $E[S|D_2]$.

It's been so long, our dice friends

$$E[X|Y=y] = \sum_x x p_{X|Y}(x|y)$$



- Roll two 6-sided dice.
- Let roll 1 be D_1, roll 2 be D_2.
- Let S = value of $D_1 + D_2$.

1. What is $E[S|D_2 = 6]$? $\frac{57}{6} = 9.5$
2. What is $E[S|D_2]$?
   - A. A function of S
   - B. A function of D_2
   - C. A number
3. Give an expression for $E[S|D_2]$.

$$E[S|D_2 = d_2] = E[D_1 + d_2|D_2 = d_2]$$

$$= \sum_{d_1} (d_1 + d_2) P(D_1 = d_1|D_2 = d_2)$$

$$= \sum_{d_1} d_1 P(D_1 = d_1) + d_2 \sum_{d_1} P(D_1 = d_1)$$

($D_1 = d_1$, $D_2 = d_2$ independent events)

$$= E[D_1] + d_2 = 3.5 + d_2$$

$$E[S|D_2] = 3.5 + D_2$$

Law of Total Expectation

Properties of conditional expectation

1. LOTUS:

$$E[g(X)|Y=y] = \sum_x g(x) p_{X|Y}(x|y)$$

2. Linearity of conditional expectation:

$$E\left[\sum_{i=1}^{n} X_i \mid Y = y\right] = \sum_{i=1}^{n} E[X_i|Y=y]$$

3. Law of total expectation:

$$E[X] = E\big[E[X|Y]\big]$$

what?

Proof of Law of Total Expectation

$E[X] = E[E[X|Y]]$

$$E[E[X|Y]] = E[g(Y)] = \sum_y P(Y = y) E[X|Y = y]$$ (LOTUS, $g(Y) = E[X|Y]$)

$$= \sum_y P(Y = y) \sum_x xP(X = x|Y = y)$$ (def of conditional expectation)

$$= \sum_y \left(\sum_x xP(X = x|Y = y) P(Y = y) \right) = \sum_y \left(\sum_x xP(X = x, Y = y) \right)$$ (chain rule)

$$= \sum_x \sum_y xP(X = x, Y = y) \quad = \sum_x x \sum_y P(X = x, Y = y)$$ (switch order of summations)

$$= \sum_x xP(X = x)$$ (marginalization)

$$= E[X]$$

Another way to compute $E[X]$

$E[X] = E[E[X|Y]]$

$$E[E[X|Y]] = \sum_y P(Y = y)E[X|Y = y] = E[X]$$

If we only have a conditional PMF of X on some discrete variable Y, we can compute $E[X]$ as follows:

1. Compute expectation of X given some value of $Y = y$
2. Repeat step 1 for all values of Y
3. Compute a weighted sum (where weights are $P(Y = y)$)

```
def recurse():
  if random.random() < 0.5:
    return 3
  return 2 + recurse()
```

Useful for analyzing recursive code.

Analyzing recursive code

$$E[X] = E\big[E[X|Y]\big] = \sum_y E[X|Y = y]P(Y = y)$$

```
def recurse():
  # equally likely values 1,2,3
  x = np.random.choice([1,2,3])
  if x == 1: return 3
  if x == 2: return 5 + recurse()
  return 7 + recurse()
```

Let Y = return value of `recurse()`.
What is $E[Y]$?

Analyzing recursive code

$$E[X] = E\big[E[X|Y]\big] = \sum_y E[X|Y=y]P(Y=y)$$

```python
def recurse():
  # equally likely values 1,2,3
  x = np.random.choice([1,2,3])
  if x == 1: return 3
  if x == 2: return 5 + recurse()
  return 7 + recurse()
```

Let Y = return value of `recurse()`.
What is $E[Y]$?

$$E[Y] = E[Y|X=1]P(X=1) + E[Y|X=2]P(X=2) + E[Y|X=3]P(X=3)$$

$$E[Y|X=1] = 3$$

When $X = 1$, return 3.

Analyzing recursive code

If Y discrete

$$E[X] = E\big[E[X|Y]\big] = \sum_y E[X|Y=y]P(Y=y)$$

```
def recurse():
  # equally likely values 1,2,3
  x = np.random.choice([1,2,3])
  if x == 1: return 3
  if x == 2: return 5 + recurse()
  return 7 + recurse()
```

Let Y = return value of `recurse()`.
What is $E[Y]$?

$$E[Y] = E[Y|X=1]P(X=1) + E[Y|X=2]P(X=2) + E[Y|X=3]P(X=3)$$

$E[Y|X=1] = 3$

What is $E[Y|X=2]$?

A. $E[5] + Y$
B. $E[5+Y] = 5 + E[Y]$
C. $5 + E[Y|X=2]$

Analyzing recursive code

If Y discrete

$$E[X] = E\big[E[X|Y]\big] = \sum_y E[X|Y = y]P(Y = y)$$

```
def recurse():
  # equally likely values 1,2,3
  x = np.random.choice([1,2,3])
  if x == 1: return 3
  if x == 2: return 5 + recurse()
  return 7 + recurse()
```

Let Y = return value of `recurse()`.
What is $E[Y]$?

$$E[Y] = E[Y|X = 1]P(X = 1) + E[Y|X = 2]P(X = 2) + E[Y|X = 3]P(X = 3)$$

$E[Y|X = 1] = 3$

When $X = 2$, return 5 + a future return value of `recurse()`.

What is $E[Y|X = 2]$?

A. $E[5] + Y$
B. $E[5 + Y] = 5 + E[Y]$
C. $5 + E[Y|X = 2]$

Analyzing recursive code

If Y discrete

$$E[X] = E\big[E[X|Y]\big] = \sum_y E[X|Y=y]P(Y=y)$$

```
def recurse():
  # equally likely values 1,2,3
  x = np.random.choice([1,2,3])
  if x == 1: return 3
  if x == 2: return 5 + recurse()
  return 7 + recurse()
```

Let Y = return value of `recurse()`.
What is $E[Y]$?

$$E[Y] = E[Y|X=1]P(X=1) + E[Y|X=2]P(X=2) + E[Y|X=3]P(X=3)$$

$E[Y|X=1] = 3$

$E[Y|X=2] = 5 + E[Y]$

When $X = 3$, return $7 +$ a future return value of `recurse()`.

$$E[Y|X=3] = E[7+Y]$$

Analyzing recursive code

If Y discrete
$$E[X] = E\big[E[X|Y]\big] = \sum_y E[X|Y=y]P(Y=y)$$

```
def recurse():
  # equally likely values 1,2,3
  x = np.random.choice([1,2,3])
  if x == 1: return 3
  if x == 2: return 5 + recurse()
  return 7 + recurse()
```

Let Y = return value of `recurse()`.
What is $E[Y]$?

$$E[Y] = E[Y|X=1]P(X=1) + E[Y|X=2]P(X=2) + E[Y|X=3]P(X=3)$$

$$E[Y|X=1] = 3 \qquad E[Y|X=2] = 5 + E[Y] \qquad E[Y|X=3] = 7 + E[Y]$$

$$E[Y] = 3(1/3) + (5+E[Y])(1/3) + (7+E[Y])(1/3)$$

$$E[Y] = (1/3)(15 + 2E[Y]) = 5 + (2/3)E[Y]$$

$$E[Y] = 15$$

On your own: What is Var(Y)?

Independent RVs, defined another way

If X and Y are **independent** discrete random variables, then $\forall x, y$:

$$P(X = x | Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)} = \frac{P(X = x)P(Y = y)}{P(Y = y)} = P(X = x)$$

$$p_{X|Y}(x|y) = \frac{p_{X,Y}(x, y)}{p_Y(y)} = \frac{p_X(x)p_Y(y)}{p_Y(y)} = p_X(x)$$

Note for conditional expectation, independent X and Y implies

$$E[X|Y = y] = \sum_x x p_{X|Y}(x|y) = \sum_x x p_X(x) = E[X]$$

Random number of random variables

indep X, Y
$E[X|Y = y] = E[X]$

Suppose you have a website: **`dorisisthebeast.com`**. Let:

- $X =$ # of people per day who visit your site. $X \sim \text{Poi}(50)$
- $Y_i =$ # of minutes spent per day by visitor i $Y_i \sim \text{Poi}(11)$
- X and all Y_i are independent.

The time spent by all visitors per day is $W = \sum_{i=1}^{X} Y_i$. What is $E[W]$?

Random number of random variables

Suppose you have a website: **`dorisisthebeast.com`**. Let:

- X = # of people per day who visit your site. $X \sim \text{Poi}(50)$
- Y_i = # of minutes spent per day by visitor i. $Y_i \sim \text{Poi}(11)$
- X and all Y_i are independent.

The time spent by all visitors per day is $W = \sum_{i=1}^{X} Y_i$. What is $E[W]$?

$$E[W] = E\left[\sum_{i=1}^{X} Y_i\right] = E\left[\; E\left[\sum_{i=1}^{X} Y_i \,|X\right] \;\right]$$

Suppose $X = x$.

$$E\left[\sum_{i}^{x} Y_i \,|X = x\right] = \sum_{i=1}^{x} E[Y_i|X = x] \qquad \text{(linearity)}$$

$$= \sum_{i=1}^{x} E[Y_i] \qquad \text{(independence)}$$

$$= xE[Y_i]$$

$$= E\big[XE[Y_i]\big]$$

$$= E[Y_i]E[X] \qquad \text{(scalar } E[Y_i]\text{)}$$

$$= 11 \cdot 50 = 550$$