Section 1: Combinatorics and Probability

Chris Piech, Mehran Sahami, Jerry Cain, Lisa Yan, and numerous CS109 CA's.

Overview of Section Materials

The warm-up questions provided will help students practice concepts introduced in lectures. The section problems are meant to apply these concepts in more complex scenarios similar to what you will see in problem sets and exams. In fact, many of them **are** old exam questions.

Before you leave lab, make sure you click here so that you're marked as having attended this week's section. The CA leading your discussion section can enter the password needed once you've submitted.

Warm-ups

1. Equality versus Inequality

Show that for any events A and B that

$$P(A) + P(B) - 1 \leq P(A \cap B) \leq P(A \cup B) \leq P(A) + P(B)$$

For each of the three inequalities, describe sets A and B that would result in equality.

2. Fish Pond

Suppose there are 7 blue fish, 4 red fish, and 8 green fish in a large fishing tank. You drop a net into it and end up with 2 fish. What is the probability you get 2 blue fish?

Problems

3. Rolling Fair Dice

Consider an experiment where we roll a fair, six-sided die multiple times.

a. What is the probability that at least one 3 appears when you roll the same fair die 10 times?

b. What is the probability that at least two 3's appear when you roll the same fair die 20 times? You may leave your answer in terms of one or more choose terms.

c. What is the probability that at least n 3's appear when you roll the same fair die $10n$ times? Your answer will certainly involve a sum of many combinatorial terms, and you needn't simplify provided we understand the structure of your answer.

d. Do you expect the probability from part c to increase or decrease as n increases? Provide some intuition as to why you expect the increase or decrease.

4. Baking Cookies

The following problem is based on true events. It was also a take-home exam question several years ago.

Jerry and the CS109 course staff are baking M&M cookies on a rainy Saturday morning, but they only have enough flour to make 6 cookies. They have 15 M&M's, all of which are different colors. For all sub-parts, assume that we don't distinguish between different arrangements of M&M's on *the same* cookie. That is, M&M's have no ordering on a cookie.

a. How many ways can the 15 M&M's be distributed across the six cookies? For this subpart, assume the cookies themselves ARE distinguishable, and the M&M's ARE distinguishable. It is possible for a cookie to have no M&M's, and it is possible for a cookie to have all of them.

b. How many ways can the 15 M&M's be distributed across the six cookies? For this subpart, assume the cookies themselves are distinguishable, and the M&M's are indistinguishable. It is possible for a cookie to have no M&M's, and it is possible for a cookie to have all of them.

c. What's the probability that each of the six cookies ends up with a different number of M&M's? Note that this would require that each of the six cookies get 0, 1, 2, 3, 4, and 5 M&M's in some order. For this subpart, assume the cookies themselves are distinguishable, and the M&M's are distinguishable. Assume further that an M&M is equally likely to appear on any cookie.

d. If we no longer require all M&M's be used, what's the probability all cookies end up with the same number of M&M's? For this subpart, assume the cookies themselves are distinguishable, and the M&M's are distinguishable. Assume further that an M&M is equally likely to appear on any cookie, and that we should include the possibility that none of the cookies get M&M's. Concretely, treat "no cookie" as another cookie with equal likelihood to other cookies.

5. The Birthday Problem

When solving a counting problem, it can often be useful to come up with a generative process, a series of steps that "generates" examples. A correct generative process to count the elements of set A will (1) *generate every element of A* and (2) *not generate any element of A more than once*. If our process has the added property that (3) *any given step always has the same number of possible outcomes*, then we can use the product rule of counting.

Problem: Assume that birthdays happen on any of the 365 days of the year with equal likelihood (we'll ignore leap years).

a. What is the probability that of the n people in class, at least two people share the same birthday?

b. What is the probability that this class contains exactly one pair of people who share a birthday?

6. Flipping Coins

One thing that students often find tricky when learning combinatorics is how to figure out when a problem involves permutations and when it involves combinations. Naturally, we will look at a problem that can be solved with both approaches. Pay attention to what parts of your solution represent distinct objects and what parts represent indistinct objects.

Problem: We flip a fair coin n times, hoping (for some reason) to get k heads.

a. How many ways are there to get exactly k heads? Characterize your answer as a *permutation* of H's and T's.

b. For what x and y is your answer to part (a) equal to $\binom{x}{y}$? Why does this *combination* make sense as an answer?

c. What is the probability that we get exactly k heads?

7. Combinatorial Proofs

Show that $\binom{m+n}{k} = \sum_{j=0}^{k} \binom{m}{j}\binom{n}{k-j}$ via a combinatorial proof.