Section 1 Solution: Combinatorics and Probability

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Overview of Section Materials

The warm-up questions provided will help students practice concepts introduced in lectures. The section problems are meant to apply these concepts in more complex scenarios similar to what you will see in problem sets and exams. In fact, many of them **are** old exam questions.

Before you leave lab, make sure you click here so that you're marked as having attended this week's section. The CA leading your discussion section can enter the password needed once you've submitted.

Warm-ups

1. Equality versus Inequality

Show that for any events A and B that

$$P(A) + P(B) - 1 \leq P(A \cap B) \leq P(A \cup B) \leq P(A) + P(B)$$

For each of the three inequalities, describe sets A and B that would result in equality.

We'll show each of the three to be true, one by one.

- The Inclusion-Exclusion Principle is clear that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. Because all probabilities, including those of intersections, are greater than or equal to 0, we have that $P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq P(A) + P(B)$. When does the equality hold? When A and B are mutually exclusive events so that the probability that both events occur is 0.
- $P(A \cap B) \leq P(A \cup B)$ is true simply because $A \cap B \subseteq A \cup B$. Equality holds when A and B are the same event.
- Inclusion-Exclusion states that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ so that $P(A \cup B) + P(A \cap B) = P(A) + P(B)$. Because all probabilities are at most 1, we have $P(A) + P(B) = P(A \cap B) + P(A \cup B) \leq P(A \cap B) + 1$, so that $P(A) + P(B) - 1 \leq P(A \cap B)$. Equality holds when the union of A and B is the full sample space S.

2. Fish Pond

Suppose there are 7 blue fish, 4 red fish, and 8 green fish in a large fishing tank. You drop a net into it and end up with 2 fish. What is the probability you get 2 blue fish?

For the full sample space, we consider the number of ways we can select 2 fish from all 19 fish without regard for blue, red, or green. We treat all of the fish as distinct in order to make sure that each event is equally likely. We don't consider order of fish to matter. The size of our sample space is thus $\binom{19}{2}$. For our event space, we consider the number of ways we can select 2 blue fish. There are 7 distinct blue fish from which we can chose two. Recall that the event space must be consistent with the sample space:

$$p = \frac{\binom{7}{2}}{\binom{19}{2}} = \frac{21}{171} \approx 0.123$$

The same answer could be arrived at by using the chain rule. The probability that the first fish is blue is 7/19. The probability that the second fish is blue, given that the first fish was blue is 6/18. The probability that both fish are blue is $7/19 \cdot 6/18 \approx 0.123$

Problems

3. Rolling Fair Dice

Consider an experiment where we roll a fair, six-sided die multiple times.

a. What is the probability that at least one 3 appears when you roll the same fair die 10 times?

Let A be the event that we roll one or more 3's in 10 rolls of the fair die. It's easier to exclude the scenario where we never get any 3's than it is to include all the ways to get one or more of them. We compute $P(A)$ by counting equally likely outcomes, where the full sample space is of size 6^{10}. The number of ways to get precisely zero 3's is 5^{10}, so that $P(A^C) = \frac{5^{10}}{6^{10}}$ and $P(A) = 1 - \frac{5^{10}}{6^{10}}$.

What is the probability that at least two 3's appear when you roll the same fair die 20 times? You may leave your answer in terms of one or more choose terms.

Let B be the event of interest this time, and once again let's compute $P(B^C)$ by counting the fraction of equally likely events that lead to 0 or 1 3's among the 20 rolls of the die. The outcome space is of size 6^{20} this time, and there are 5^{20} ways to be devoid of 3's, and $\binom{20}{1}5^{19}$ ways to generate exactly one 3 somewhere within 19 other non-3's. That means that:

$$P(B^C) = \frac{5^{20} + \binom{20}{1}5^{19}}{6^{20}}$$

$$P(B) = 1 - \frac{5^{20} + \binom{20}{1}5^{19}}{6^{20}}$$

b. What is the probability that at least n 3's appear when you roll the same fair die $10n$ times? Your answer will certainly involve a sum of many combinatorial terms, and you needn't simplify provided we understand the structure of your answer.

We'll go with D this time, since C already means event complement.

$$P(D^C) = \frac{\sum_{k=0}^{n-1} \binom{10n}{k} 5^{10n-k}}{6^{10n}}$$

$$P(D) = 1 - \frac{\sum_{k=0}^{n-1} \binom{10n}{k} 5^{10n-k}}{6^{10n}}$$

c. Do you expect the probability from part c to increase or decrease as n increases? Provide some intuition as to why you expect the increase or decrease.

We should expect the probability to increase an n gets larger and larger and approach 1.0. Intuitively, as n grows towards infinity, we should expect the number of 3's to be roughly $\frac{1}{6}$ (or about 16.6%) of all rolls, so the probability that the number of 3's be less than 10% of all rolls should eventually vanish as farcically improbable.

4. Baking Cookies

The following problem is based on true events. It was also a take-home exam question several years ago.

Jerry and the CS109 course staff are baking M&M cookies on a rainy Saturday morning, but they only have enough flour to make 6 cookies. They have 15 M&M's, all of which are different colors. For all sub-parts, assume that we don't distinguish between different arrangements of M&M's on *the same* cookie. That is, M&M's have no ordering on a cookie.

a. How many ways can the 15 M&M's be distributed across the six cookies? For this subpart, assume the cookies themselves ARE distinguishable, and the M&M's ARE distinguishable. It is possible for a cookie to have no M&M's, and it is possible for a cookie to have all of them.

$$\boxed{6^{15}}$$

There are 6 places to put the first M&M, 6 places to put the second M&M, ... all the way to 15. Using the product rule, this gives us $6 * 6 * \ldots$

b. How many ways can the 15 M&M's be distributed across the six cookies? For this subpart, assume the cookies themselves are distinguishable, and the M&M's are indistinguishable. It is possible for a cookie to have no M&M's, and it is possible for a cookie to have all of them.

$$\boxed{\binom{15+6-1}{6-1}}$$

We use the divider method because the "items" (M&M's) are indistinguishable and the amounts we can put in each "bucket" (cookie) are not fixed. We have 15 items and 6 buckets, so $6 - 1$ dividers.

c. What's the probability that each of the six cookies ends up with a different number of M&M's? Note that this would require that each of the six cookies get 0, 1, 2, 3, 4, and 5 M&M's in some order. For this subpart, assume the cookies themselves are distinguishable, and the M&M's are distinguishable. Assume further that an M&M is equally likely to appear on any cookie.

$$\frac{|E|}{|S|} = \boxed{\frac{6!\binom{15}{1,2,3,4,5}}{6^{15}}}$$

It would also be acceptable for the multinomial coefficient to be $\binom{15}{0,1,2,3,4,5}$. Since all the outcomes of the sample space are equally likely, the probability of an event is $\frac{|E|}{|S|}$.

$|S|$ can be computed as 6^{15} because there are 6 places to put the first M&M, 6 places to put the second M&M, ... all the way to 15. Using the product rule, this gives us $6 * 6 * \ldots$.

$|E|$ can be computed as the product of

- 6! because this accounts for the permutations of the literal numbers 0, 1, 2, 3, 4, 5
- $\binom{15}{1,2,3,4,5}$ because we are counting all the ways to put distinguishable items into groups of sizes 0, 1, 2, 3, 4, 5.

d. If we no longer require all M&M's be used, what's the probability all cookies end up with the same number of M&M's? For this subpart, assume the cookies themselves are distinguishable, and the M&M's are distinguishable. Assume further that an M&M is equally likely to appear on any cookie, and that we should include the possibility that none of the cookies get M&M's. Concretely, treat "no cookie" as another cookie with equal likelihood to other cookies.

$$\boxed{\frac{1}{7^{15}} + \frac{\frac{15!}{(15-6)!}}{7^{15}} + \frac{\frac{15!}{(15-12)!} * \frac{1}{(2!)^6}}{7^{15}}}$$

$$\begin{aligned} P(\text{all have same number}) &= P(\text{all have } 0 \cup \text{all have } 1 \cup \text{all have } 2) \\ &= P(\text{all } 0) + P(\text{all } 1) + P(\text{all } 2) \qquad \text{(Axiom 3, lecture 3)} \\ &= \frac{1}{7^{15}} + \frac{\frac{15!}{(15-6)!}}{7^{15}} + \frac{\frac{15!}{(15-12)!} * \frac{1}{(2!)^6}}{7^{15}} \end{aligned}$$

The sample space is 7^{15} (as opposed to 6^{15}) because we assign the unused M&M's to an imaginary seventh cookie we never bake.

- The event space for $P(\text{all } 0)$ computes to just one outcome (no M&M's on any cookie)
- The event space for $P(\text{all } 1)$ computes to $\frac{15!}{(15-6)!}$ because the first cookie has 15 M&M's to choose from, the second cookie has 14 M&M's, the third has 13 M&M's, etc. The remaining 9 go into UNUSED.
- The event space for $P(\text{all } 2)$ computes to the product of $\frac{15!}{(15-12)!}$ (because the first slot on the first cookie has 15 M&M's to choose from, the second slot on the first cookie has 14 M&M's, the first slot on the second cookie has 13 M&M's, etc.) and $\frac{1}{(2!)^6}$ because we want to discount different arrangements on the same cookie.

5. The Birthday Problem

When solving a counting problem, it can often be useful to come up with a generative process, a series of steps that "generates" examples. A correct generative process to count the elements of set A will (1) *generate every element of A* and (2) *not generate any element of A more than once*. If our process has the added property that (3) *any given step always has the same number of possible outcomes*, then we can use the product rule of counting.

Problem: Assume that birthdays happen on any of the 365 days of the year with equal likelihood (we'll ignore leap years).

a. What is the probability that of the n people in class, at least two people share the same birthday?

Computing $P(\text{at least 2 people share a birthday})$ is difficult. We realize that this can be thought of as

$$P(\text{exactly 2 people share birthday} \cup \text{exactly 3} \cup \text{exactly 4} \cup \ldots \cup \text{exactly } n \text{ people share birthday})$$

Using the additivity axiom of probability, we realize that this can be split up because the events are mutually exclusive.

$$P(\text{exactly 2 people share birthday}) + P(\text{exactly 3}) + P(\text{exactly 4}) + \ldots + P(\text{exactly } n \text{ people share birthday})$$

However, this is very tedious!

It is much easier to calculate $1 - P(\text{no one shares a birthday})$. Let our sample space, S be the set of all possible assignments of birthdays to the students in section. By the assumptions of this problem, each of those assignments is equally likely, so this is a good choice of sample space. We can use the product rule of counting to calculate $|S|$:

$$|S| = (365)^n$$

Our event E will be the set of assignments in which there are no matches (i.e. everyone has a different birthday). We can think of this as a generative process where there are 365 choices of birthdays for the first student, 364 for the second (since it can't be the same birthday as the first student), and so on. Verify for yourself that this process satisfies the three conditions listed above. We can then use the product rule of counting:

$$|E| = (365) \cdot (364) \cdot \cdots \cdot (365 - n + 1)$$

$$\begin{aligned} P(\text{birthday match}) &= 1 - P(\text{no matches}) \\ &= 1 - \frac{|E|}{|S|} \\ &= 1 - \frac{(365) \cdot (364) \ldots (365 - n + 1)}{(365)^n} \end{aligned}$$

A common misconception is that the size of the event E can be computed as $|E| = \binom{365}{n}$ by choosing n distinct birthdays from 365 options. However, outcomes in this event (n unordered distinct dates) cannot recreate any outcomes in the sample space $|S| = 365^n$ (n distinct dates, one for each distinct person). However if we compute the size of event E as $|E| = \binom{365}{n} n!$ (equivalent to the number above), then we can assign the n birthdays to each person in a way consistent with the sample space. The expression $\binom{365}{n} n!$ is equivalent to $\frac{365!}{(365-n)!}$ which is known as a "falling factorial" and also as "365 permute n" outside of this class.

Interesting values: $(n = 13 : p \approx 0.19)$, $(n = 23 : p \approx 0.5)$, $(n = 70 : p \geq 0.99)$.

b. What is the probability that this class contains exactly one pair of people who share a birthday?

We can use the same sample space, but our event is a little bit trickier. Now E is the set of birthday assignments in which exactly two students share a birthday and the rest have different birthdays. One generative process that works for this is (1) choose the two students who share a birthday, (2) choose $n-1$ birthdays in the same manner as in part a (i.e. one for the pair of students and one for each of the remaining students). We then have:

$$P(\text{exactly one match}) = \frac{|E|}{|S|} = \frac{\binom{n}{2}(365)\cdot(364)\cdot\ldots\cdot(365-n+2)}{(365)^n}$$

Many other generative processes work for this problem. Try to think of some other ones and make sure you get the same answer!

6. Flipping Coins

One thing that students often find tricky when learning combinatorics is how to figure out when a problem involves permutations and when it involves combinations. Naturally, we will look at a problem that can be solved with both approaches. Pay attention to what parts of your solution represent distinct objects and what parts represent indistinct objects.

Problem: We flip a fair coin n times, hoping (for some reason) to get k heads.

a. How many ways are there to get exactly k heads? Characterize your answer as a *permutation* of H's and T's.

We want to know the number of sequences of n H's and T's such that there are k H's and $n-k$ T's. This is the same as permuting n objects of which one set of k is indistinguishable and one set of $n-k$ is indistinguishable. Using our formula for the permutation of indistinguishable objects, we get $\frac{n!}{k!(n-k)!}$

b. For what x and y is your answer to part (a) equal to $\binom{x}{y}$? Why does this *combination* make sense as an answer?

Our answer to part a is equal to $\binom{n}{k}$. This makes sense because we can come up with a valid sequence by *choosing* k flips to come out to heads (and implicitly define the other $n-k$ to be tails). The answer is also equivalent to $\binom{n}{n-k}$ for which the same logic applies except with choosing flips to be tails.

c. What is the probability that we get exactly k heads?

If we define our sample space to be all possible sequences of flips, then our event is the number of sequences where we get exactly k heads, meaning that $|E|$ is (conveniently) the answer to the previous two parts. Our probability is then $\frac{|E|}{|S|} = \frac{\binom{n}{k}}{2^n}$.

7. Combinatorial Proofs

Show that $\binom{m+n}{k} = \sum_{j=0}^{k}\binom{m}{j}\binom{n}{k-j}$ via a combinatorial proof.

A fully algebraic proof requires fluency in combinatorics beyond the level that you'll see in this class. However, we can prove the above by relying on a combinatorial argument!

The left hand side counts the number of ways I can form a committee of size k from the combined junior and senior high school classes, each of size m and n respectively. The right hand side says the committee can be formed by choosing k of the m juniors and no seniors, $k - 1$ of the m juniors and just 1 of the n seniors, $k - 2$ juniors and and 2 seniors, and so forth. In general, we can form the committee by selecting j juniors in any one of $\binom{m}{j}$ ways, and for each of those selections, we can independently fill out the rest of the committee by selecting $k - j$ seniors in any one of $\binom{n}{k-j}$ ways, for all reasonable values of j.