17: Continuous Joint Distributions II

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May 8th, 2024

Lecture Discussion on Ed

Convolution: Sum of independent Uniform RVs

Today's lecture

Take what we've seen with discrete joint distributions…

…and generalize to continuous joint distributions.

For the most part, this isn't too bad. Examples:

	Discrete	Continuous (densities)
Marginal distributions	$p_X(a) = \sum_y p_{X,Y}(a,y)$	$f_X(a) = \int_{-\infty}^{\infty} f_{X,Y}(a,y)dy$
Independent RVs	$p_{X,Y}(x,y) = p_X(x)p_Y(y)$	$f_{X,Y}(x,y) = f_X(x)f_Y(y)$

CDFs: $\rightarrow F_{X,Y}(x,y) = F_X(x)F_Y(y)$

But some concepts, while mathematically accessible given what we've learned, are difficult to implement in practice.

We'll focus on some of these today.

> Goal of CS109 continuous joint distributions unit: **build mathematical maturity**

Dance, Dance, Convolution

Review

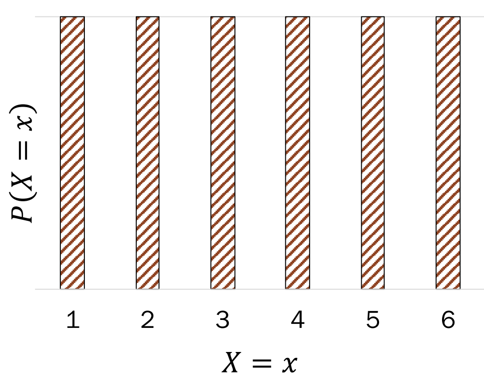
Recall that for independent discrete random variables X and Y:

$$P(X+Y=n)=\sum_k P(X=k)P(Y=n-k)$$

the convolution of p_X and p_Y

$$p_{X+Y}(n)=\sum_k p_X(k)\,p_Y(n-k)$$

same thing, more compact



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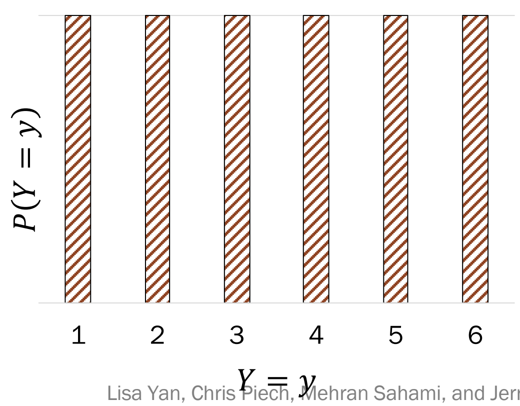

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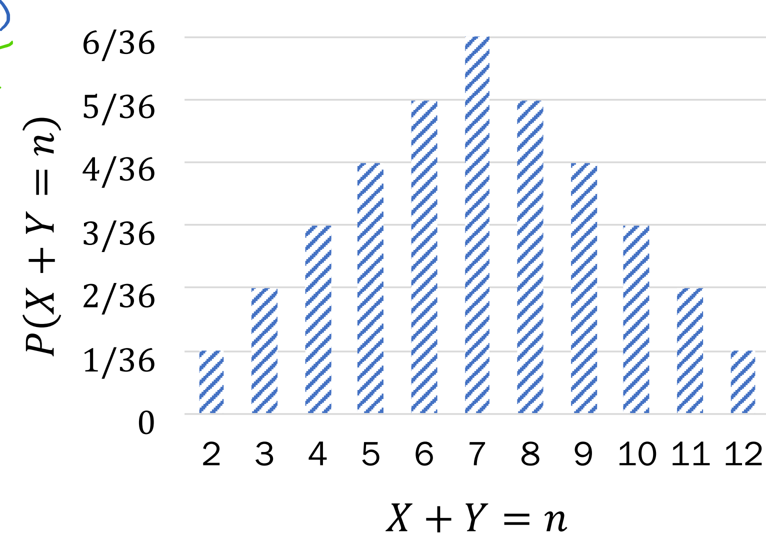


Independent X, Y

Dance, Dance, Convolution

Recall that for independent discrete random variables X and Y:

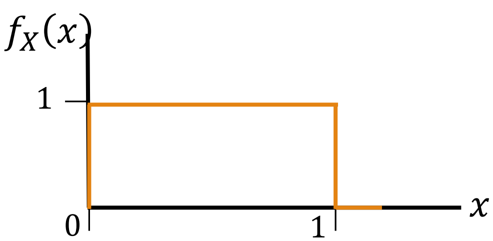
$$P(X+Y=n) = \sum_k P(X=k)P(Y=n-k)$$

the convolution of p_X and p_Y

For independent **continuous** random variables X and Y:

$$f_{X+Y}(\alpha) = \int_{-\infty}^{\infty} f_X(x) f_Y(\alpha - x)\,dx$$

the convolution of f_X and f_Y



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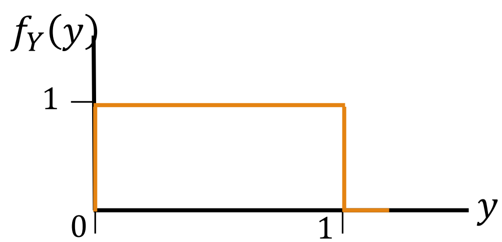

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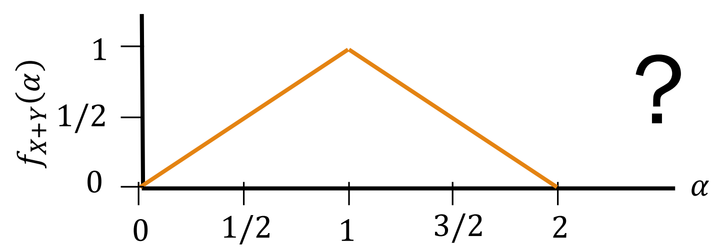


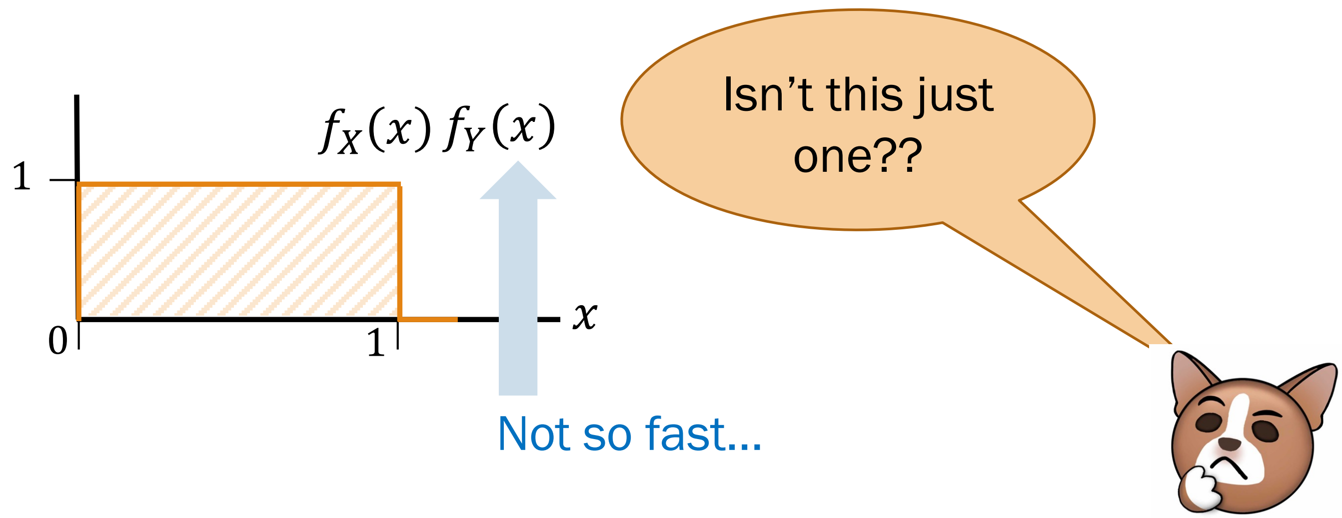
?

Independent X, Y

Sum of independent Uniforms

Let $X \sim \text{Uni}(0,1)$ and $Y \sim \text{Uni}(0,1)$ be independent RVs.
What is the distribution of $X + Y$, $f_{X+Y}(\alpha)$?

$$f_{X+Y}(\alpha) = \int_{-\infty}^{\infty} f_X(x) f_Y(\alpha - x) dx$$



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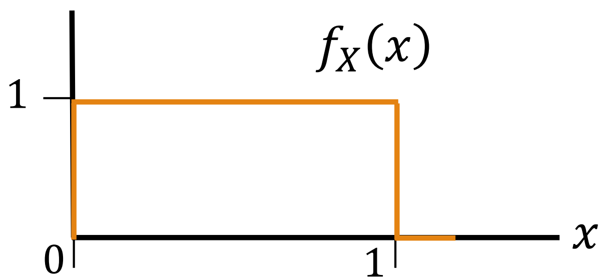

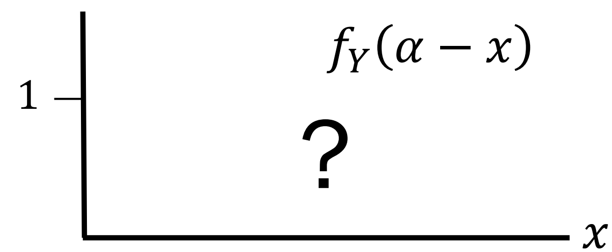

$$f_X(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(\alpha - x) = \begin{cases} 1 & \text{if } 0 \le \alpha - x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} 1 & \text{if } \alpha - 1 \le x \le \alpha \\ 0 & \text{otherwise} \end{cases}$$

1.) subtract α from everything, then,
2.) divide by -1

α is a constant in the integral w.r.t. x.

Sum of independent Uniforms

X and Y independent + continuous $\quad f_{X+Y}(\alpha) = \int_{-\infty}^{\infty} f_X(x) f_Y(\alpha - x)\, dx$

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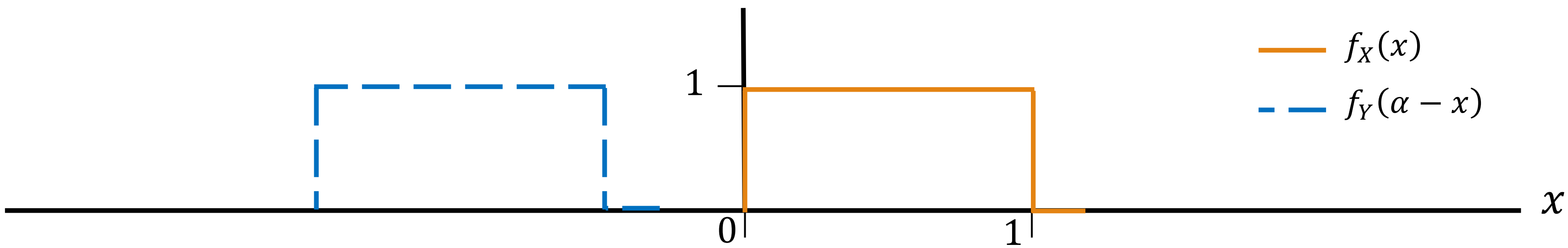

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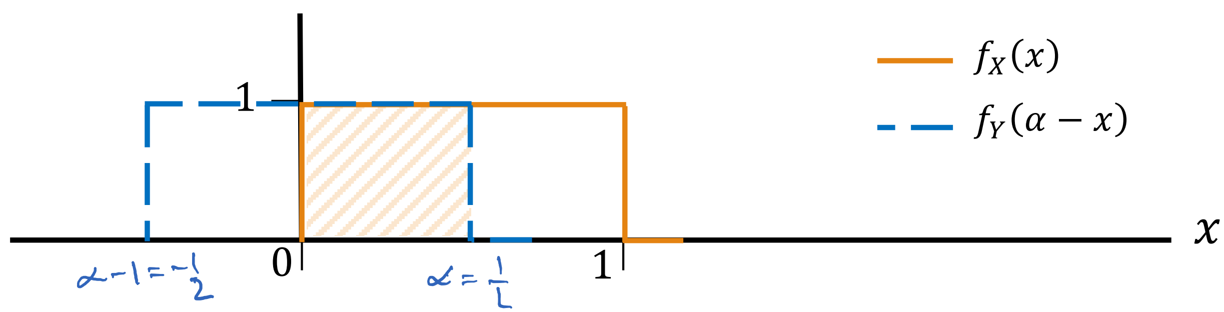

Integral = area under the curve
This curve = product of 2 functions of x

Sum of independent Uniforms

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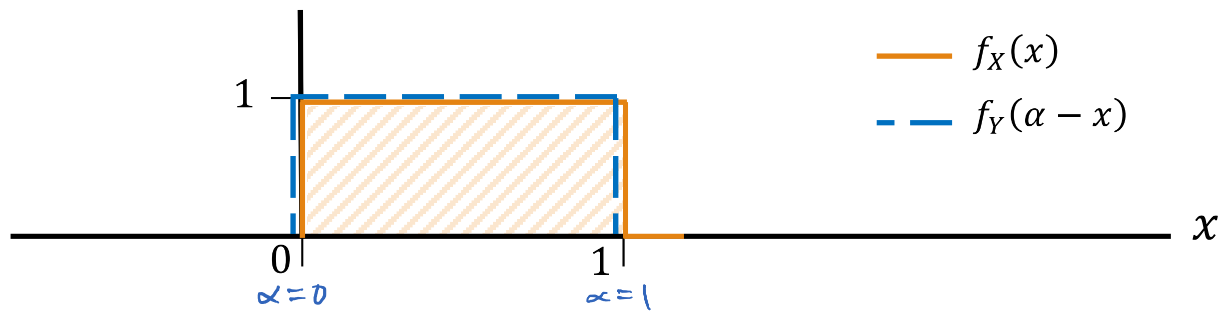

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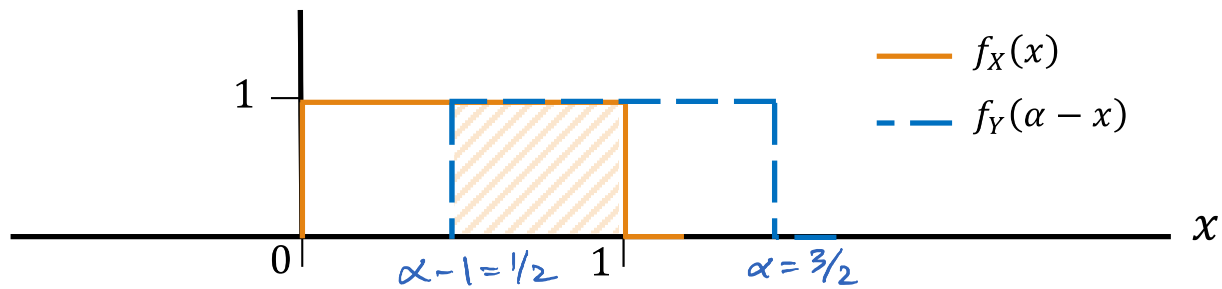

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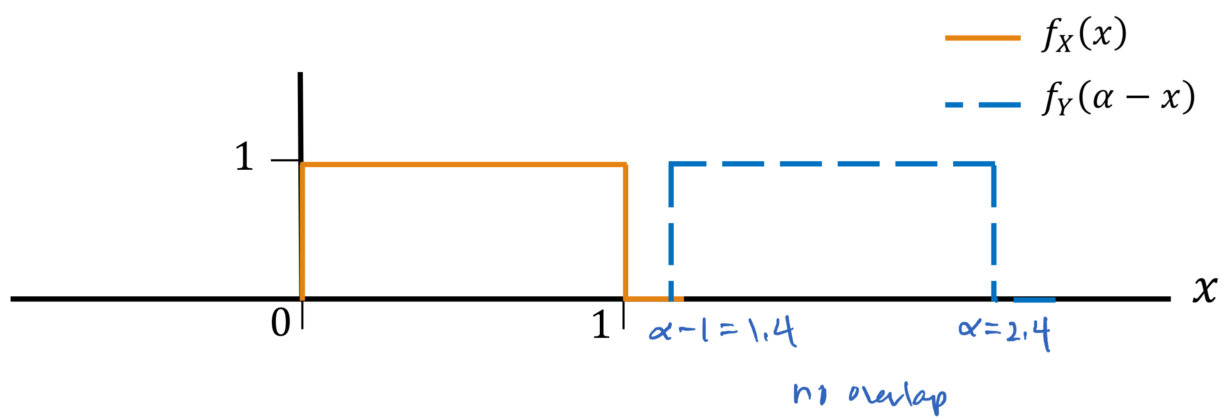

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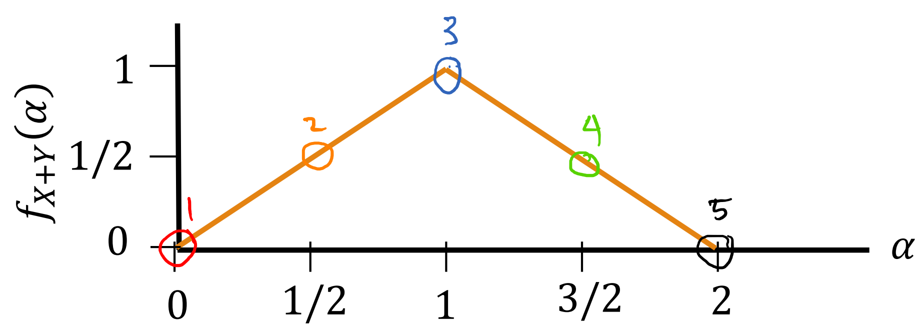
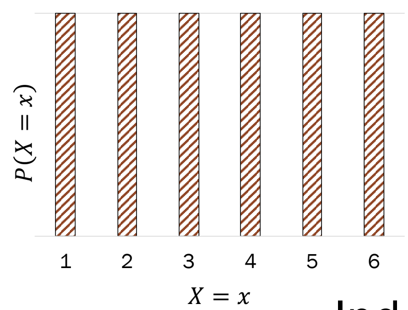


$$f_{X+Y}(\alpha) = \begin{cases} \alpha & 0 \le \alpha \le 1 \\ 2 - \alpha & 1 \le \alpha \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Dance, Dance, Convolution Extreme



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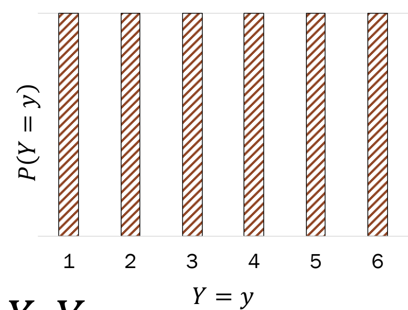

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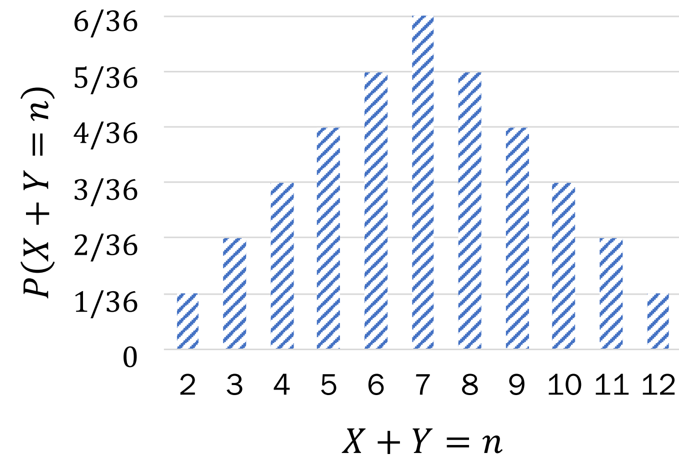
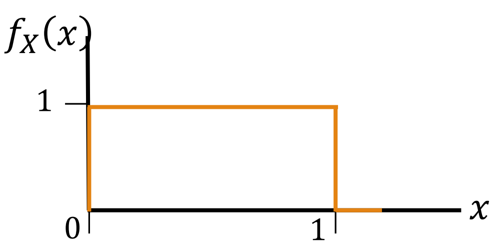


Independent X, Y



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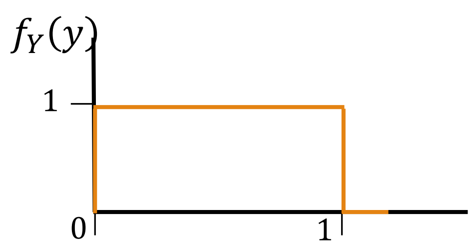

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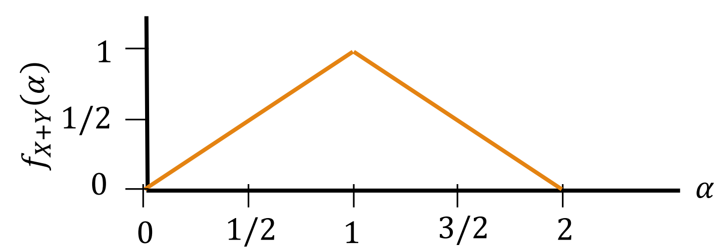


Independent X, Y

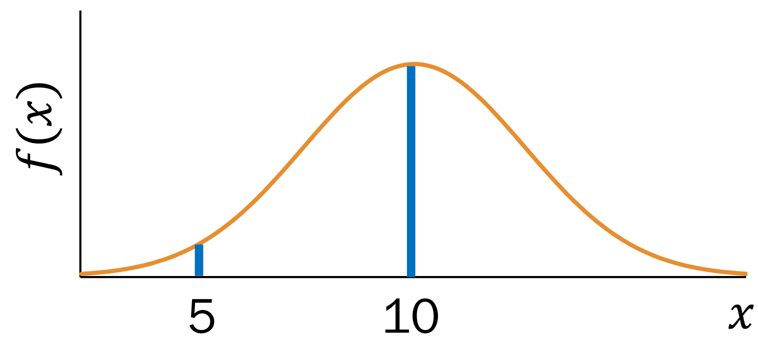
Ratio of PDFs

Relative probabilities of continuous random variables

Let X = time to finish Problem Set 4.

Suppose $X \sim \mathcal{N}(10, 2)$.

How much **more likely** are you to complete in 10 hours than 5 hours?



$$\frac{P(X = 10)}{P(X = 5)} =$$

A. 0/0 = undefined

B. 2

C. $\frac{f(10)}{f(5)}$

D. $\frac{f(2)}{f(1)}$

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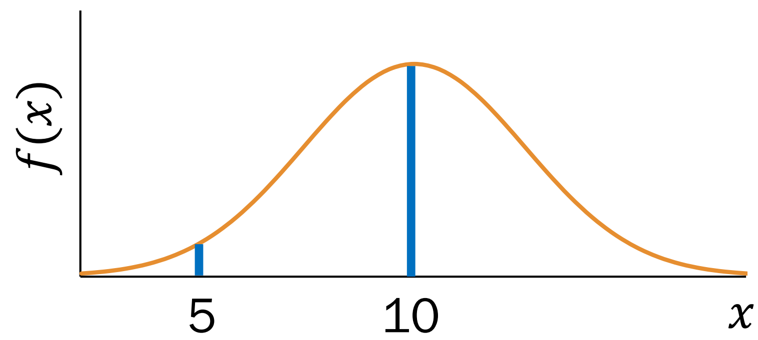

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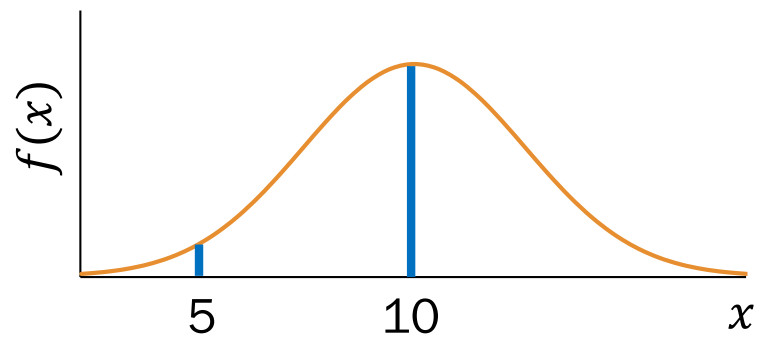


$$\frac{P(X=10)}{P(X=5)} = \frac{f(10)}{f(5)}$$

$$P(X=a) = P\left(a - \frac{\varepsilon}{2} \le X \le a + \frac{\varepsilon}{2}\right) = \int_{a-\frac{\varepsilon}{2}}^{a+\frac{\varepsilon}{2}} f(x)dx \approx \varepsilon f(a)$$

Therefore $\dfrac{P(X=a)}{P(X=b)} = \dfrac{\varepsilon f(a)}{\varepsilon f(b)} = \dfrac{f(a)}{f(b)}$

$$= \frac{\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(10-\mu)^2}{2\sigma^2}}}{\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(5-\mu)^2}{2\sigma^2}}} = \frac{e^{-\frac{(10-10)^2}{2\cdot 2}}}{e^{-\frac{(5-10)^2}{2\cdot 2}}} = \frac{e^0}{e^{-\frac{25}{4}}} = 518$$

Ratios of PDFs are meaningful!

Continuous conditional distributions

Continuous conditional distributions

For continuous RVs X and Y, the **conditional PDF** of X given Y is

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

where $f_Y(y) > 0$

Intuition: $P(X = x|Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)}$ ⟺ $f_{X|Y}(x|y)\varepsilon_X = \frac{f_{X,Y}(x,y)\varepsilon_X\varepsilon_Y}{f_Y(y)\varepsilon_Y}$

Note that conditional PDF $f_{X|Y}$ is a "true" density:

$$\int_{-\infty}^{\infty} f_{X|Y}(x|y)dx = \int_{-\infty}^{\infty} \frac{f_{X,Y}(x,y)}{f_Y(y)}dx = \frac{f_Y(y)}{f_Y(y)} = 1$$

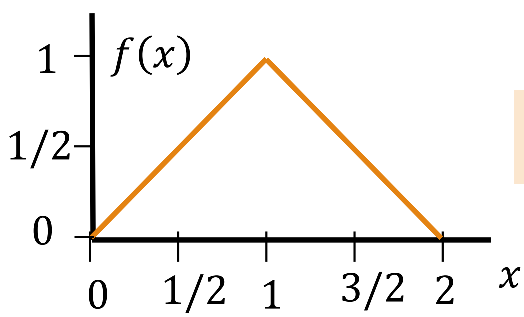
Why sums of random variables?

Sometimes modeling and understanding a complex RV, X, is difficult.

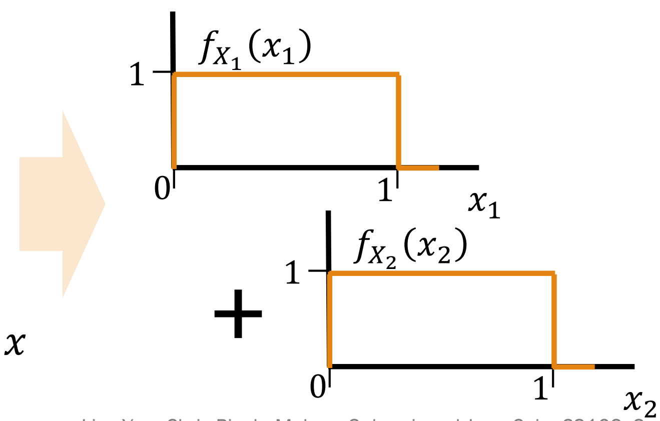
But if we can decompose X into the **sum of simpler, independent** RVs,

- We can compute distributions on X.
- We can better understand how X changes as its constituent RVs change.

What can we model with a triangular PDF?



Sum of uniforms!



We're covering the reverse direction for now; the forward direction will come on Friday

Everything* in probability is a sum or a product (or both)

*except conditional probability (a ratio)

Sum of values that can be considered separately (possibly weighted by prob. of happening)

$$E[X] = \sum_x \underbrace{xp(x)}_{\text{weight}}$$

$$E[X|Y=y] = \int_{-\infty}^{\infty} x \underbrace{f_{X|Y}(x|y)}_{\text{weight}} dx$$

$$P(E) = \sum_{i=1}^{n} P(E|F_i)\underbrace{P(F_i)}_{\text{weight}}$$

Law of Total Probability

$$P(E) = \sum_{i=1}^{n} P(E_i)$$

Axiom 3, $E = E_1 \cup \cdots \cup E_n$

assuming mutual exclusion.

Product of values that can each be considered in sequence

$$P(E \cap F \cap G) = P(E)P(F|E)P(G|EF)$$

Chain Rule

$$f_{X,Y}(x,y) = f_X(x)f_Y(y)$$

Independent cont. RVs

$$P(X+Y=n) = \sum_k P(X=k)P(Y=n-k)$$

Sum of indep. discrete RVs (convolution)

Conditional probability and Bayes' Theorem

Definition

$$P(F|E) = \frac{P(E \cap F)}{P(E)}$$

Scaling to the correct sample space

Independence

E, F independent

$$P(F|E) = P(F)$$

Sample space doesn't need to be scaled

Bayes' Theorem

$$P(F|E) = \frac{P(F)P(E|F)}{P(E)}$$

Prior: some prob. of event F

Likelihood

Posterior: prob. of F knowing that E happened

Scaling to the correct sample space

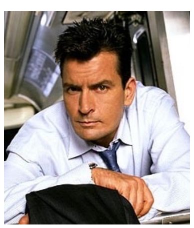
Multiple Bayes' Theorems

with
events

$$P(F|E) = \frac{P(F)P(E|F)}{P(E)}$$

with
discrete RVs

$$p_{Y|X}(y|x) = \frac{p_Y(y)p_{X|Y}(x|y)}{p_X(x)}$$



with
continuous RVs

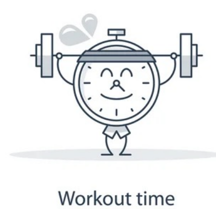
You are given this value... (x)

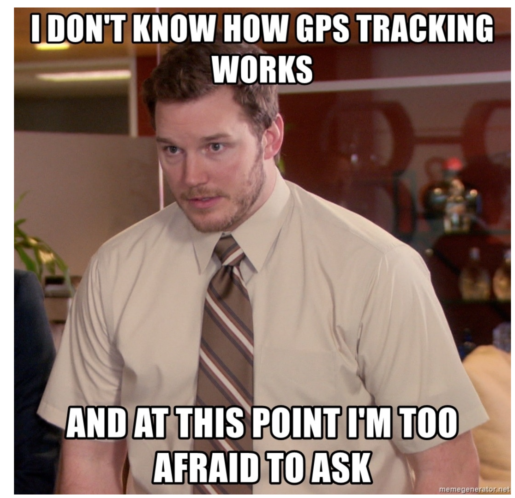
$$f_{Y|X}(y|x) = \frac{f_Y(y)f_{X|Y}(x|y)}{f_X(x)}$$

...so this is just a scalar ($f_X(x)$)

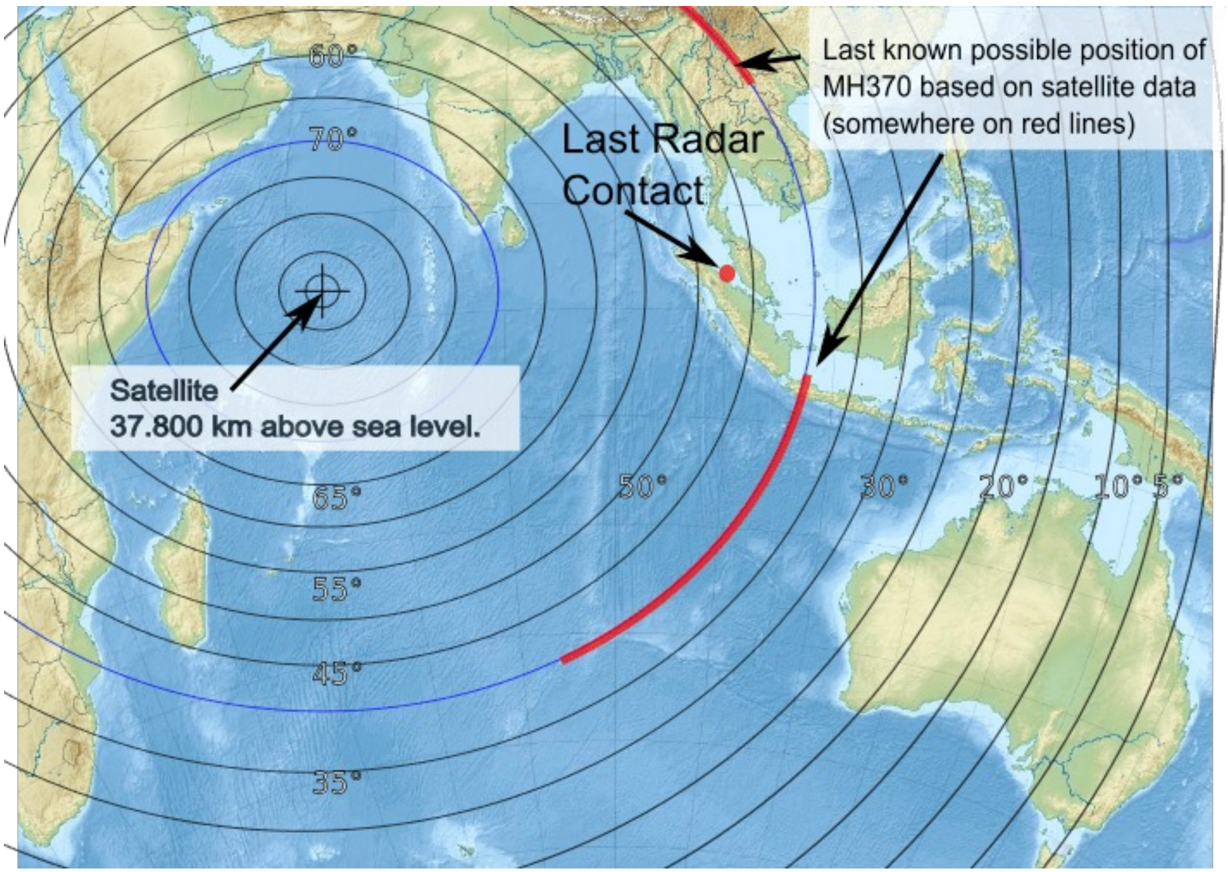
Really all the same idea!

Intense Exercise





Tracking in 2-D space



You want to know the 2-D location of an object.

Your satellite ping gives you a noisy 1-D measurement of the distance of the object from the satellite (0,0).

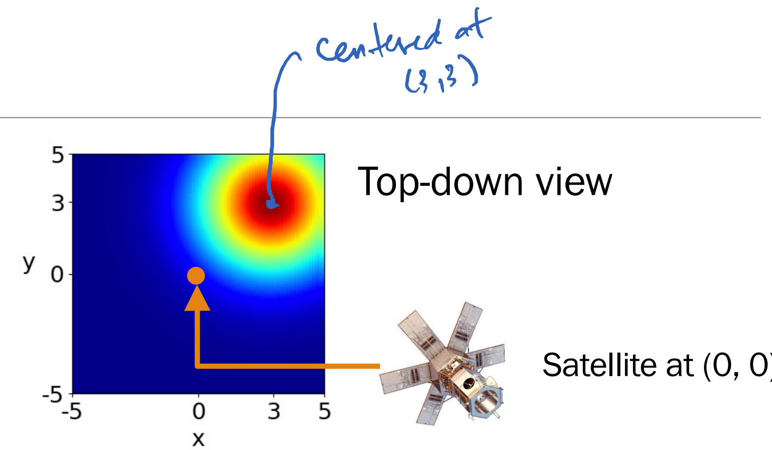
Using the satellite measurement, where is the object?

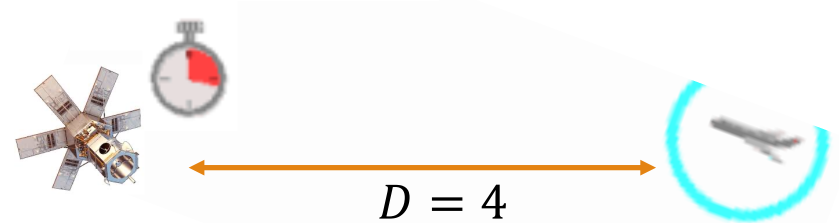
Tracking in 2-D space

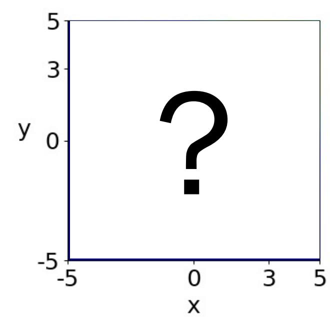
- Before measuring, we have some **prior belief** about the 2-D location of an object, (X, Y).

- We observe some noisy **measurement** $D = 4$, the Euclidean distance of the object to a satellite.

- After the measurement, what is our **updated (posterior) belief** of the 2-D location of the object?







Tracking in 2-D space

- You hold some prior beliefs about the 2-D location of an object, (X, Y).
- You observe a noisy distance measurement, $D = 4$.
- How do you update your beliefs about the 2-D location of the object after that noisy measurement?

Recall Bayes terminology:

posterior belief

likelihood (of evidence)

prior belief

$$f_{X,Y|D}(x, y|d) = \frac{f_{D|X,Y}(d|x, y) f_{X,Y}(x, y)}{f_D(d)}$$

normalization constant

1. Define prior

$$f_{X,Y|D}(x,y|d) = \frac{f_{D|X,Y}(d|x,y)\, f_{X,Y}(x,y)}{f_D(d)}$$

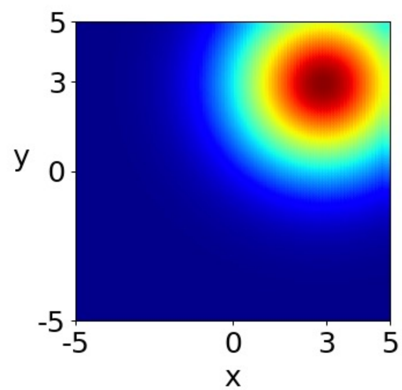
You have a prior belief about the 2-D location of an object, (X, Y).

Let (X, Y) = object's 2-D location, assuming satellite is at (0,0)

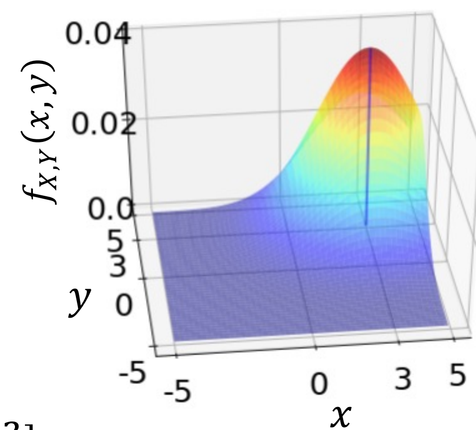
Suppose the prior distribution is a symmetric bivariate normal distribution:

this means that both marginals are normals.

Top-down view



3-D view



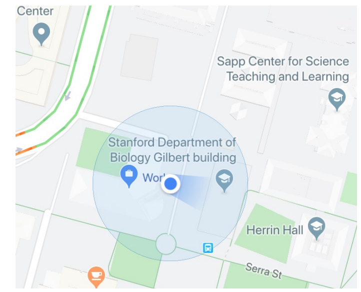
$$f_{X,Y}(x,y) = \frac{1}{2\pi 2^2} e^{-\frac{[(x-3)^2+(y-3)^2]}{2(2^2)}} = K_1 \cdot e^{-\frac{[(x-3)^2+(y-3)^2]}{8}}$$

normalizing constant

we do this because, as you'll see, we don't always care about the exact value of constants.

2. Define likelihood

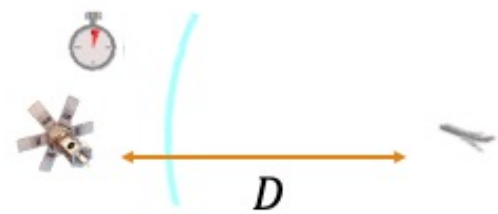
$$f_{X,Y|D}(x,y|d) = \frac{f_{D|X,Y}(d|x,y)\, f_{X,Y}(x,y)}{f_D(d)}$$



You observe a noisy distance measurement, $D = 4$.

If you knew your actual location to be (x, y), you could argue Just how likely a measurement of $D = 4$ actually is.

Let $D =$ measured radial distance from the satellite, where actual (x, y) is known!



- D is still noisy! Suppose noise is **standard normal.**
- On average, D is your true Euclidean distance: $\sqrt{x^2 + y^2}$

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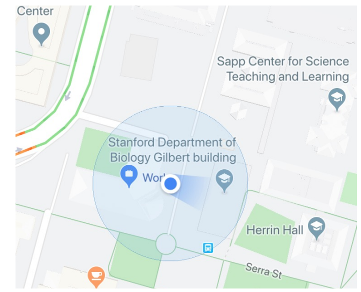

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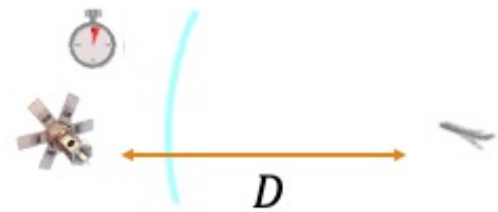


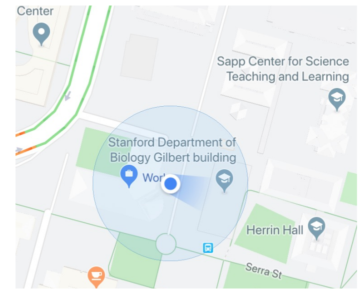
- D is still noisy! Suppose noise is **standard normal.**
- On average, D is your true Euclidean distance: $\sqrt{x^2 + y^2}$

$$D|X,Y \sim N\left(\mu = \quad (A) \quad , \sigma^2 = (B)\right)$$

$$f_{D|X,Y}(D = d|X = x, Y = y) = \frac{1}{(C)} e^{-\,(D)}$$

2. Define likelihood

$$f_{X,Y|D}(x,y|d) = \frac{\boxed{f_{D|X,Y}(d|x,y)}\, f_{X,Y}(x,y)}{f_D(d)}$$



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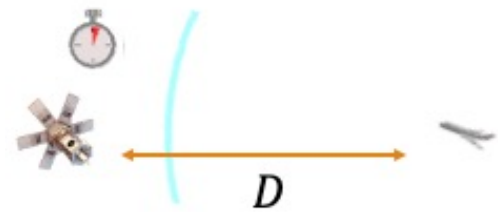


- D is still noisy! Suppose noise is **standard normal.**
- On average, D is your true Euclidean distance: $\sqrt{x^2 + y^2}$

$$D|X,Y \sim N\left(\mu = \sqrt{x^2 + y^2}, \sigma^2 = 1\right)$$

$$f_{D|X,Y}(D = d|X = x, Y = y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(d - \sqrt{x^2+y^2}\right)^2} = K_2 \cdot e^{-\frac{1}{2}\left(d - \sqrt{x^2+y^2}\right)^2}$$

normalizing constant

3. Compute posterior

$$\boxed{f_{X,Y|D}(x,y|d)} = \frac{f_{D|X,Y}(d|x,y)\ f_{X,Y}(x,y)}{f_D(d)}$$

What is your updated (posterior) belief of the 2-D location of the object after observing the measurement?

Compute:

Posterior belief

$$f_{X,Y|D}(x,y|4) = f_{X,Y|D}(X=x, Y=y|D=4)$$

3. Compute posterior

$$f_{X,Y|D}(x,y|d) = \frac{f_{D|X,Y}(d|x,y)\ f_{X,Y}(x,y)}{f_D(d)}$$

What is your updated (posterior) belief of the 2-D location of the object after observing the measurement?

Compute:

Posterior belief

$$f_{X,Y|D}(x,y|4) = f_{X,Y|D}(X=x,Y=y|D=4)$$

Know:

Prior belief

$$f_{X,Y}(x,y) = K_1 \cdot e^{-\frac{[(x-3)^2+(y-3)^2]}{8}}$$

Observation likelihood

$$f_{D|X,Y}(d|x,y) = K_2 \cdot e^{-\frac{1}{2}\left(d-\sqrt{x^2+y^2}\right)^2}$$

Tips
- Use Bayes' Theorem!
- $f_D(4)$ is just a scaling constant. Why?
- How can we approximate the final scaling constant with a computer?

Tracking in 2-D space

What is your updated (posterior) belief of the 2-D location of the object after observing the measurement?

likelihood of $D = 4$ prior belief

$$f_{X,Y|D}(X = x, Y = y|D = 4) = \frac{f_{D|X,Y}(D = 4|X = x, Y = y) f_{X,Y}(x, y)}{f(D = 4)}$$

Bayes' Theorem

$$= \frac{K_2 \cdot e^{-\frac{\left(4-\sqrt{x^2+y^2}\right)^2}{2}} \cdot K_1 \cdot e^{-\frac{[(x-3)^2+(y-3)^2]}{8}}}{f(D = 4)}$$

$$= \frac{K_3 \cdot e^{-\left[\frac{\left(4-\sqrt{x^2+y^2}\right)^2}{2} + \frac{[(x-3)^2+(y-3)^2]}{8}\right]}}{f(D = 4)}$$

$$= K_4 \cdot e^{-\left[\frac{\left(4-\sqrt{x^2+y^2}\right)^2}{2} + \frac{[(x-3)^2+(y-3)^2]}{8}\right]}$$

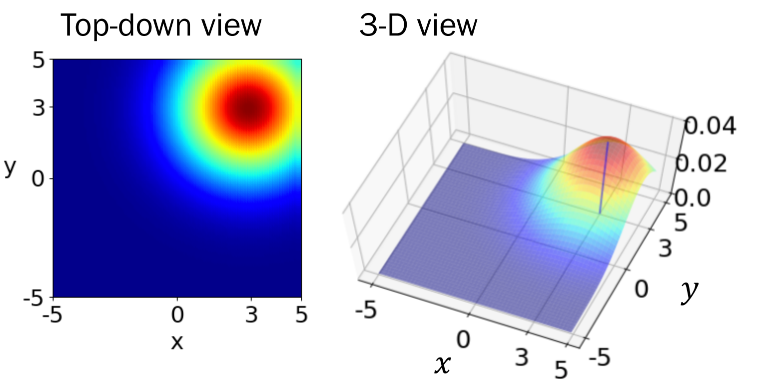
Key: Once we know the part dependent on x, y, we can computationally approximate K_4 so that $f_{X,Y|D}$ is a valid PDF.

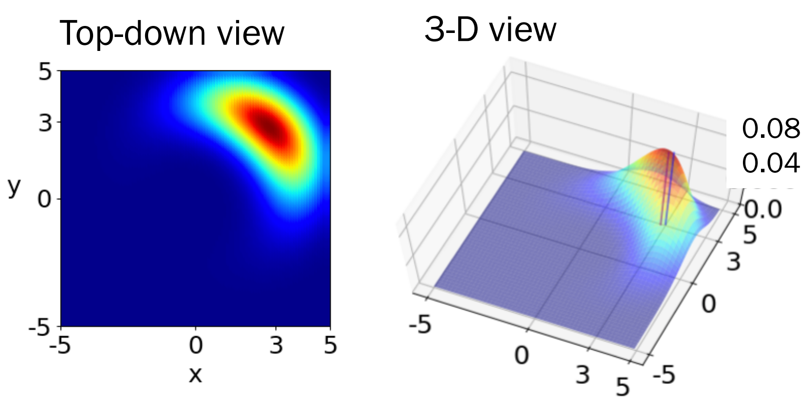
Tracking in 2-D space

(3,3) is at 4.2 units from origin

With this continuous version of Bayes' theorem, we can explore new domains.

- Before measuring, you hold some prior beliefs about the 2-D location of an object, (X, Y).
- You observe a noisy distance measurement, D=4.
- After the measurement, do you update your beliefs about the 2-D location of the object after that noisy measurement.

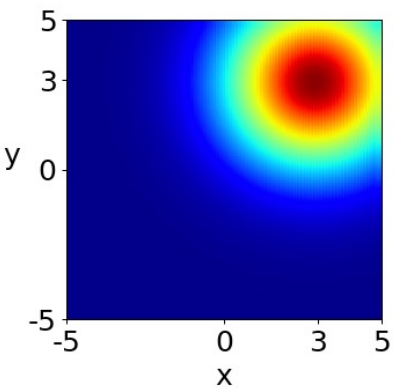




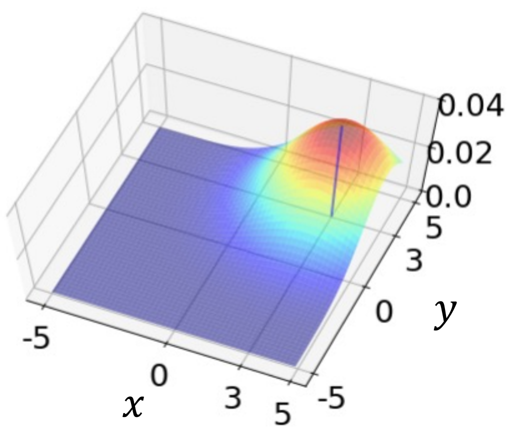
Tracking in 2-D space: Posterior belief

Prior belief

Top-down view



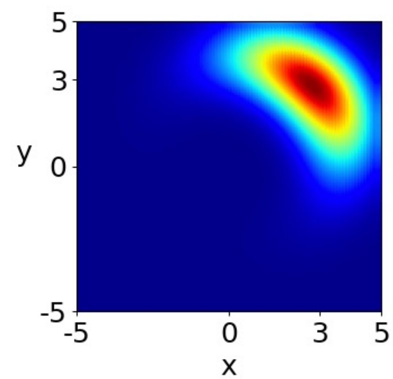
3-D view



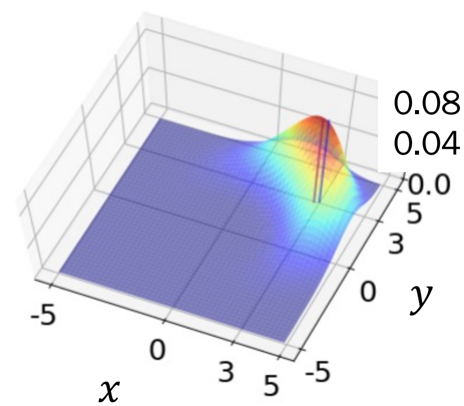
$$f_{X,Y}(x,y) = K_1 \cdot e^{-\frac{[(x-3)^2+(y-3)^2]}{8}}$$

Posterior belief

Top-down view



3-D view



$$f_{X,Y|D}(x,y|4) =$$

$$K_4 \cdot e^{-\left[\frac{\left(4-\sqrt{x^2+y^2}\right)^2}{2}+\frac{[(x-3)^2+(y-3)^2]}{8}\right]}$$

How'd you compute that K_4?

To be a valid conditional PDF, $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y|D}(x,y|4)\,dx\,dy = 1$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_4 \cdot e^{-\left[\frac{\left(4-\sqrt{x^2+y^2}\right)^2}{2}+\frac{\left[(x-3)^2+(y-3)^2\right]}{8}\right]} dx\,dy = 1$$

$$\frac{1}{K_4} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\left[\frac{\left(4-\sqrt{x^2+y^2}\right)^2}{2}+\frac{\left[(x-3)^2+(y-3)^2\right]}{8}\right]} dx\,dy$$

(pull out K_4, divide)

Approximate:

$$\frac{1}{K_4} \approx \sum_{y}\sum_{x} e^{-\left[\frac{\left(4-\sqrt{x^2+y^2}\right)^2}{2}+\frac{\left[(x-3)^2+(y-3)^2\right]}{8}\right]} \Delta x \Delta y$$

Use a computer!